(Symmetry protected) topological order

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Contents

Literature

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Part I

Spin chains and symmetry protected topological order

1 Path integral of a spin

1.1 Spin coherent states

Consider[1] a spin s particle. We define the spin coherent state as

$$\hat{\vec{S}} \cdot \hat{n} \, |\hat{n}\rangle = s \, |\hat{n}\rangle \, , \tag{1}$$

where $\hat{n} \in \mathbb{S}^2$ is a unit vector.

Comments

- In spherical coordinates $\hat{n} = \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}$ with $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$.

We now explicitly construct such a state

- Let $s = 1/2$ and by consequence $\hat{\vec{S}} = \vec{\sigma}/2$. We easily see that a state sufficing Eq. (1) can be constructed by using a projector

$$|\hat{n}\rangle_{s=1/2} \equiv \mathcal{N} \frac{1 + \hat{n} \cdot \vec{\sigma}}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\phi/2} \cos(\theta/2) \\ e^{i\phi/2} \sin(\theta/2) \end{pmatrix} \equiv z. \tag{2}$$

  In the definition, we included a normalization constant, which turns out to be $\mathcal{N} = e^{-i\phi/2}/\cos(\theta/2)$. Exploiting the Clifford algebra of Pauli matrices we also conclude that Eq. (1) implies for an arbitrary function $f(\hat{\vec{S}})$ that

$$\langle \hat{n} | f(\hat{\vec{S}}) | \hat{n} \rangle_{s=1/2} = \langle \hat{n} | f(\hat{n}/2) | \hat{n} \rangle_{s=1/2} \, . \tag{3}$$

[1] This chapter follows the lecture notes on “quantum condensed matter physics” by C. Nayak.

– One may calculate the overlap by calculating the absolute square value

$$\langle \hat{n} | \hat{n}' \rangle_{s=1/2} = e^{i\alpha(\hat{n},\hat{n}')} \sqrt{\frac{1 + \hat{n} \cdot \hat{n}'}{2}}, \tag{4}$$

where $\alpha(\hat{n}, \hat{n}') = -\alpha(\hat{n}', \hat{n})$ is a phase. We will only need its value for $\hat{n}'$ close to $\hat{n}$, for which it is

$$e^{i\alpha(\hat{n},\hat{n}')} \simeq 1 - i\cos(\theta)\frac{(\phi' - \phi)}{2} + \ldots, \tag{5}$$

where "…" are higher orders in differences of angles.

– And finally, one may use the normalized measure on the sphere ($\int_{\mathbb{S}^2} d\hat{n} = 1$) to calculate the completeness relation

$$\begin{aligned} &2 \int_{\mathbb{S}^2} d\hat{n}\, |\hat{n}\rangle_{s=1/2} \;{}_{s=1/2}\langle \hat{n}| \\ &= \frac{2}{4\pi} \int_0^{\pi} d\theta \sin(\theta) \int_0^{2\pi} d\phi \begin{pmatrix} \cos(\theta/2)^2 & e^{-i\phi} \cos(\theta/2)\sin(\theta/2) \\ e^{i\phi} \cos(\theta/2)\sin(\theta/2) & \sin(\theta/2)^2 \end{pmatrix} \\ &= \mathbf{1}. \end{aligned} \tag{6}$$

- For more general s, we simply symmetrize $2s$ spin 1/2 states

$$|\hat{n}\rangle \equiv z^{\alpha_1} \ldots z^{\alpha_{2s}} |\alpha_1, \ldots, \alpha_{2s}\rangle. \tag{7}$$

– Then the overlap and expectation value for any function $f(\hat{\vec{S}})$ are

$$\langle \hat{n} | \hat{n}' \rangle = \prod_{i=1}^{2s} z^{*\alpha_i} z'^{\alpha_i} = e^{i2s\alpha(\hat{n},\hat{n}')} \left(\frac{1 + \hat{n} \cdot \hat{n}'}{2} \right)^s \tag{8}$$

$$\langle \hat{n} | f(\hat{\vec{S}}) | \hat{n} \rangle = z^{\alpha_1,*} \ldots z^{\alpha_{2s},*} z^{\alpha'_1} \ldots z^{\alpha'_{2s}} \langle \alpha_1 \ldots \alpha_{2s} | f(\sum_{i=1}^{2s} \frac{\vec{\sigma}_i}{2}) | \alpha'_1 \ldots \alpha'_{2s} \rangle = f(s\hat{n}) \tag{9}$$

– The completeness relation in the $2s + 1$ dimensional Hilbert space

$$\mathbf{1} = (2s + 1) \int_{\mathbb{S}^2} d\hat{n}\, |\hat{n}\rangle \langle \hat{n}| \tag{10}$$

follows from two ingredients:

∗ The proportionality to unity of the operator on the right is due to Schur's lemma, i.e. that $\hat{S}_i |\hat{n}\rangle \langle \hat{n}| = s\hat{n}_i |\hat{n}\rangle \langle \hat{n}| = |\hat{n}\rangle \langle \hat{n}| \hat{S}_i$, $\forall i$.

∗ The normalization follows from taking the trace on either side.

1.2 Path integral

Using the spin coherent states we can derive the path integral for a spin using the standard Trotterization formula

$$\begin{aligned}\mathcal{Z} &= \operatorname{tr} e^{-\beta H(\hat{\vec{S}})} \\ &= \lim_{N\to\infty} \int \prod_{i=1}^{N} [(2s+1)d\hat{n}_i] \prod_{i=1}^{N} \underbrace{\langle \hat{n}_{i+1}| e^{-\Delta\tau H(\hat{\vec{S}})} |\hat{n}_i\rangle}_{\simeq \exp[(\langle \hat{n}_{i+1}|\hat{n}_i\rangle - 1) - \Delta\tau H(s\hat{n})]} \\ &= \int \mathcal{D}n \exp[-\int_0^\beta d\tau \, \langle \hat{n}|\partial_\tau \hat{n}\rangle + H(s\hat{n})] \end{aligned} \tag{11}$$

In the Trotterization we used the convention $\Delta\tau = \beta/N$ and found a term $\langle \hat{n}|\partial_\tau \hat{n}\rangle$ corresponding to the kinetic term in coherent state path integrals for fermions or bosons. We now study this term further and use Eq. (8) and use that $\alpha(\hat{n}, \hat{n}')$ is an odd function under exchange of its two arguments variables and thus leads to a linear coefficient in the expansion

$$\alpha(\hat{n}(\tau - \Delta\tau/2), \hat{n}(\tau + \Delta\tau/2)) = -i\Delta\tau\dot{\phi}\cos(\theta)/2, \tag{12}$$

while $\hat{n}(\tau + \Delta/2) \cdot \hat{n}(\tau = \Delta/2) \sim 1 + \mathcal{O}(\Delta\tau^2)$. Thus we find $S = S_H + S_{\mathrm{WZ}}$ with

$$\begin{aligned} S_H &= \int_0^\beta d\tau H(s\hat{n}) \\ S_{\mathrm{WZ}} &= -is \int_0^\beta d\tau \cos(\theta)\dot{\phi} = is \int_0^\beta d\tau \dot{\hat{n}} \cdot \vec{A}, \end{aligned} \tag{13}$$

where we used $\dot{\hat{n}} = \dot{\theta}\hat{e}_\theta + \dot{\phi}\sin(\theta)\hat{e}_\phi$ and

$$\vec{A} = \frac{1-\cos(\theta)}{\sin(\theta)} \underbrace{\begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix}}_{\hat{e}_\phi} \Rightarrow \nabla \times \vec{A} = \hat{e}_r. \tag{14}$$

Thus, the action formalism for a spin-s is equivalent to the action formalism of a particle of charge s on a sphere with a magnetic monopole of strength 4π in the center.

Using this and Stokes' theorem we find

$$S_{\mathrm{WZ}} = is \int_{\gamma=\partial S} d\hat{n} \cdot \vec{A}(\hat{n}) \overset{\text{Stokes}}{=} is \int_S d^2S \; \hat{e}_r \cdot (\nabla \times \vec{A}) = is \; \mathrm{area}(S). \tag{15}$$

However, there is an ambiguity in this construction, as γ may either be seen as a curve enclosing an area around the north pole or an area enclosing the south pole. As the two shouldn't differ, we have to enforce

$$2\pi i\mathbb{Z} \ni is\left[\int_{S_{north}} d^2S\ \hat{e}_r\cdot(\nabla\times\vec{\mathcal{A}}) - \underbrace{(-1)}_{\substack{\text{reverse}\\\text{orientiation}}}\int_{S_{south}} d^2S\ \hat{e}_r\cdot(\nabla\times\vec{\mathcal{A}})\right] = is\int_{\mathbb{S}^2} d^2S = i4\pi s. \tag{16}$$

Importantly, only the quantization of angular momentum in half integers, i.e. $s\in\mathbb{Z}/2$, leads to consistent theories.

1.3 General summary of topological terms

In the above construction, we have seen that for the 0D field theory of a spin, a topological term can occur. Here we generalize this concept and summarize which terms can appear in D dimensional field theories on an arbitrary closed manifold $\mathcal{M}$ (e.g. manifold of Goldstone modes).

- **Wess-Zumino-Novikov-Witten (WZW) terms**
  - rely on a non-trivial $D+1$th homotopy group $\pi_{D+1}(\mathcal{M})=\mathbb{Z}$.
  - The Action takes the form

    $$S_{\text{WZW}} = i2\pi k\Gamma[Q],$$

    where $Q:\mathbb{R}^D\to\mathcal{M}$ and
    * $\Gamma[Q]$ measures the normalized solid hyper-angle enclosed in a (hyper-)loop.
    * As a consequence that closing the solid angle into nord-pole or south pole is arbitrary and $\Gamma_{\text{North}}[Q]-\Gamma_{\text{South}}[Q]\in\mathbb{Z}$, it follows that the "Wess-Zumino-level" $k\in\mathbb{Z}$ has to be quantized (otherwise the partition function would be ill-defined).
- **$\mathbb{Z}$ theta terms in a D-dimensional system.**
  - rely on a non-trivial Dth homotopy group $\pi_D(\mathcal{M})=\mathbb{Z}$.
  - A simple example occurs for the quantum mechanics of a particle on ring (see exercise)
  - The action takes the form

    $$S_\theta = i\theta N[Q], \tag{17}$$

    where

    * $N[Q] \in \mathbb{Z}$ is quantized and counts the (winding-)number of instantons in Q and thereby affects the way different topological sectors are weighted in the partition sum.
    * While θ in general is not quantized.
  - If the "angle" $\theta = \pi(2n+1), n \in \mathbb{Z}$, the model is critical, as the sectors with even and odd number of instantons destructively interfere. .

- **Other terms, e.g. $\mathbb{Z}_2$ theta terms**
  - In principle, other topological terms may also appear.
  - For example, $\mathbb{Z}_2$ theta terms rely on $\pi_D(\mathcal{M}) = \mathbb{Z}_2$.
  - Similarly to the $\mathbb{Z}$ theta terms,

$$S_\theta = i\theta N[Q], \tag{18}$$

  where
    * $N[Q] \in \mathbb{Z}_2$ is quantized and counts the (winding-)number of instantons in Q and thereby affects the way different topological sectors are weighted in the partition sum.
    * however, more like in the WZW case, for the $\mathbb{Z}_2$ case $\theta = 0, \pi$ is also quantized.
  - It can often be understood as descendant of the WZW term. For example, For the spin-s, this could happen when the motion is constrained to the equator (easy plane problem). Then, both prefactor and integral are quantized.

2 Antiferromagnetic chain and Haldane's conjecture

We will now consider an antiferromagnetic chain

$$H = J \sum_i \hat{\vec{S}}_i \cdot \hat{\vec{S}}_{i+1}, \tag{19}$$

which is thus described by the action

$$S = \int_0^\beta d\tau \sum_i \left[\frac{Js^2}{2} (\hat{n}_i + \hat{n}_{i+1})^2 + is\dot{\hat{n}} \cdot \vec{A} \right]. \tag{20}$$

- In the limit $s \to \infty$, we can consider the saddle point solutions: clearly the static Néel state minimizes this action, while the ferromagnet maximizes it.

- We will now derive effective action of Goldstone modes around the N'eel minimum $\hat{n}_i \simeq (-1)^i \hat{d}(x_i)$.
- To do so, we also keep small fluctuations controlled by s near the momentum of the ferromagnet so actually

$$\hat{n}_i \simeq (-1)^i \hat{d}(x_i) + \frac{1}{s}\vec{l}(x_i) \tag{21}$$

where both functions $\hat{d}(x)$ and $\vec{l}(x)$ are are slow on the scale of the lattice constant, $\hat{d}$ is a unit vector and we assume $\vec{l} \cdot \hat{d} = 0$ to enforce $\hat{n}^2 = 1 + \mathcal{O}(1/s^2)$.

- From the Hamiltonian contribution to the action we obtain

$$S_{\rm H} \simeq \int d\tau dx \frac{Js^2 a}{2}(\partial_x \hat{d})^2 + \frac{2J}{a}\vec{l}^2 \tag{22}$$

- For the topological term $S_{WZ} = is \int d\tau \dot{\hat{n}} \cdot \vec{A}$ we use to order $1/s$

$$\begin{aligned} \dot{\hat{n}}_i \cdot \vec{A}(\hat{n}_i) &\simeq (-1)^i \dot{\hat{d}} \cdot \vec{A}((-1)^i \hat{d}) + \frac{1}{s}\left(\dot{l}_a A_a((-1)^i \hat{d}) + (-1)^i \dot{\hat{d}}_a \underbrace{\partial_b A_a((-1)^i \hat{d})}_{=(-1)^i \epsilon_{bac}(\hat{e}_r)_c + \partial_a A_b} l_b \right) \\ &= \underbrace{(-1)^i \dot{\hat{d}} \cdot \vec{A}((-1)^i \hat{d})}_{\to S_{\rm WZ}[(-1)^i \hat{d}]} + \underbrace{\frac{1}{s}\partial_\tau(\vec{l} \cdot \vec{A}((-1)^i \hat{d}))}_{\to 0} + \frac{1}{s}\vec{l} \cdot (\dot{\hat{d}} \times \hat{d}) \end{aligned} \tag{23}$$

We next use $S_{\rm WZ}[(-1)^i \hat{d}] = (-1)^i S_{\rm WZ}[\hat{d}] \mathrm{mod} 4\pi$ and we can repeat the previous calculation using $\vec{l}/s \to a\partial_x \hat{d}$ to obtain

$$\begin{aligned} \sum_i S_{\rm WZ}[(-1)^i \hat{d}(x_i)] &= is \sum_{i \text{ odd}} \int d\tau [\hat{d}(x_i + a) - \hat{d}(x_i)](\dot{\hat{d}} \times \hat{d}) \\ &\to i\frac{s}{2} \int dx d\tau \hat{d}(\hat{d}' \times \dot{\hat{d}}) \end{aligned} \tag{24}$$

Here we assumed periodic boundary conditions and note that $\sum_{i \text{ odd}} \to \frac{1}{2a}\int dx$.

In total we thus find

$$\begin{aligned} S &= \int d\tau dx \quad i\frac{\vec{l} \cdot (\dot{\hat{d}} \times \hat{d})}{a} + \frac{Js^2 a}{2}(\partial_x \hat{d})^2 + \frac{2J}{a}\vec{l}^2 + S_\theta, \\ &\overset{\int \mathcal{D}\vec{l}}{\to} \frac{K}{2} \int d\tau dx \frac{1}{v}(\partial_\tau \hat{d})^2 + v(\partial_x \hat{d})^2 + S_\theta. \end{aligned} \tag{25}$$

Comments:

- This effective low-energy theory is the major result of this section. It is the NLσM describing the Goldstone modes
- The magnon speed is $v = 2asJ$ and the stiffness $K = s/2$.
- The topological term has the property

$$S_\theta = i\frac{s}{2}\int dxd\tau \hat{d}(\hat{d}' \times \dot{\hat{d}}) \\ = i2\pi s N[\hat{d}] \tag{26}$$

  where $N[\hat{d}]$ measures the skyrmion number in the field configuration. (For a picture of a skyrmion, see Fig. 1)
- Thus we found a θ term for the 1 + 1 dimensional field integral with θ angle $\theta = 2\pi s$.
  * Clearly, S_θ has no impact on the bulk partition function for integer spin. By Mermin Wagner's theorem, the model is guantum disordered and the spectrum of the quantum model has a gap $\Delta \sim Je^{-\#K}$ at lowest energies. (This statement can for example be found using renormalization group around the $K = \infty$ fixed point).
  * In contrast, for s half-integer, S_θ can not be disregarded, as it leads to destructive interference of field configurations with even/odd number of skyrmions.
  * Indeed it is known from Bethe's 1931 exact results ("Bethe Ansatz") that the spin-1/2 chain is gapless.
  * The last two points led to "Haldane's conjecture", namely that the bulk partition function of integer spin systems is gapped, while the Bulk partition function of half-integer spin systems is critical.

3 Spin-1 chain with open boundary conditions and Definition of SPT phases

Let's consider the spin-s chain with s integer. We have already established that the bulk is gapped. As we know that the bulk is gapped it is reasonable to integrate all those modes out. Technically, one could just cut the chain in pieces of the size of one coherence length $\xi \sim e^K$ and couple the rotors by nearest neighbor interaction

$$S - S_\theta \simeq \frac{K}{2}\int d\tau \sum_i \frac{\xi}{v}(\partial_\tau \hat{d}_i) - \frac{v}{2\xi}\hat{d}_i\hat{d}_{i+1}. \tag{27}$$

At energies below Kv/ξ, the second term can be dropped.

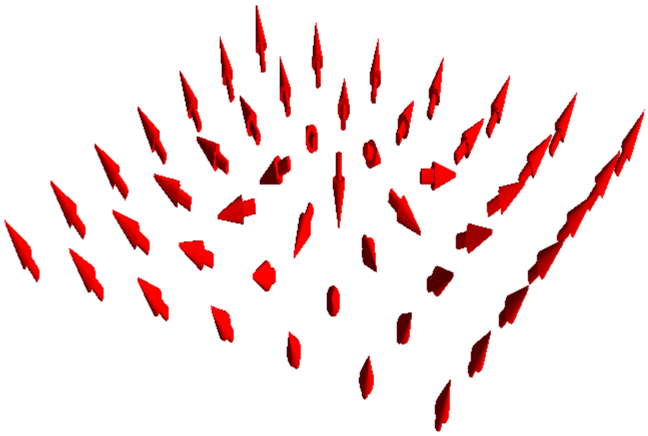


Figure 1: Graphical representation of a Skyrmion.

But what about the boundary? We can use spherical coordinates to find

$$\begin{aligned} S_\theta &= i\frac{s}{2}\int_0^\beta d\tau \int_0^L \partial_x[\cos(\theta)\dot{\phi}] \\ &= i\frac{s}{2}\int d\tau \cos(\theta)\dot{\phi}|_{x=L} - i\frac{s}{2}\int d\tau \cos(\theta)\dot{\phi}|_{x=0}. \end{aligned} \tag{28}$$

Thus, for the spin-s antiferromagnet we found

- A gap in the bulk.
- Thus at a given boundary we're left with

$$S = \frac{K\xi}{2v}\int d\tau \dot{\hat{d}}^2 + i\frac{s}{2}\int d\tau \cos(\theta)\dot{\phi}. \tag{29}$$

  The first term (the kinetic term) is RG irrelevant. The ground state physics is determined by the WZ term at level $s/2$ corresponding to a $s/2$ spin

- In particular, we expect the spin-1 chain to have spin-1/2 edge states! This is a characteristic feature of the "Haldane"-gap of the spin-1 chain and the paradigmatic example for a symmetry protected topological (SPT) phase. (WARNING: Despite Eq. (29), only odd-integer spin-chains are actually in an SPT phase, see details in the next section.)

4 Definition of symmetry protected topological order

The precise wording of this definition is taken from *Z. Bi et al. PRB **91**134404 (2015).*

Symmetry protected topological (SPT) phases should satisfy at least one of the following criteria

- *i* *On a d-dimensional lattice without boundary, this phase is fully gapped, and nondegenerate.*
- *ii* *On a d-dimensional lattice with a (d-1)-dimensional boundary, if the Hamiltonian of the entire system (including both bulk and boundary Hamiltonian) preserves certain symmetry G this phase is either gapless, or gapped but degenerate.*
- *iii* *The boundary state of this d-dimensional system cannot be realized as a (d-1)-dimensional lattice system with the same symmetry G.*

Comments:

- While different SPT phases can not be adiabatically (i.e. without gap closing) transformed into one another while preserving all symmetries, in contrast to topological order (second part of the lecture course), such adiabatic transformation do exist if the symmetries are allowed to be broken on the way.
- As such, the SPT phases are "protected" by some symmetry group G.
- There can be bosonic and fermionic SPT phases and the non-interacting topological insulator phases presented by Andreas also fall into the category of SPT.
- The classification of SPTs can occur either via classifying projective symmetry representations of edge states (XG Wen *et al.*) or using NLσM. Generalizing the field theory of the spin-1 chain in $d = 1$, it was shown by Z. Bi *et al.* PRB **91**134404 (2015) that for the most important symmetry groups G, all SPT phases in $d = 2, 3$ can be described by a NLσM with field $\hat{d} \in \mathbb{S}^{d+2}$ with θ term.
- While we naively saw that any integer spin s chain leads to the appearance of a spin $s/2$ excitations (and thus to degeneracy $s+1$) at the boundary, in fact only $s \in 2\mathbb{Z}+1$ leads to a proper SPT phase. To see this, consider the following arguments:
  - Couple spin-1 chains with term $S_{\text{coupling}} = A \int d\tau dx \hat{n}_1 \hat{n}_2$.
  - The edge would either form a low-lying singlet (for antiferromagnetic coupling) or triplet corresponding either to the same edge state as the spin $s = 0$ or spin $s = 2$ chain.

– However, as A can be tuned from negative to positive without closing the bulk gap nor breaking the protecting spin-rotation symmetry.

– Thus, it turns out that spin s and spin $s+2$ chains are equivalent.

5 AKLT model

In the previous sections, we gave some general field theoretical arguments for the existence of SPT phases using methods which essentially perturb about the large spin $s \to \infty$ limit.

In this section, we introduce a model with $s = 1$ which has an exactly solvable groundstate and displays $s = 1/2$ edge states. This model is called the AKLT (Affleck-Kennedy-Lieb-Tasaki) model and has the following Hamiltonian

$$H_{\text{AKLT}} = J \sum_i \underbrace{\frac{1}{3} + \frac{1}{2}\hat{\vec{S}}_i \cdot \hat{\vec{S}}_{i+1} + \frac{1}{6}(\hat{\vec{S}}_i \cdot \hat{\vec{S}}_{i+1})^2}_{P_{i,i+1}} \tag{30}$$

Comments

- Because of the spin-1 nature of the spins $\hat{\vec{S}}$ the biquadratic term with prefactor 1/6 is not just a unit matrix.
- With a bit of algebra one realizes

$$P_{i,i+1} = \frac{1}{24}\hat{\vec{S}}^2_{\text{tot}}[\hat{\vec{S}}^2_{\text{tot}} - 2] = \hat{P}_{S_{\text{tot}}=2}. \tag{31}$$

  so it projects onto the state with total angular momentum $\hat{\vec{S}}_{\text{tot}} = \hat{\vec{S}}_i + \hat{\vec{S}}_{i+1}$ such that $\hat{\vec{S}}^2_{\text{tot}} = S_{\text{tot}}(S_{\text{tot}} + 1)$ where $S_{\text{tot}} = 0, 1, 2$.
- The Hamiltonian is thus the sum over projectors $P_{i,i+1}$ and thus positive semi-definite and the ground state is the zero mode of the Hamiltonian.

$$H_{\text{AKLT}} \left|GS\right\rangle = 0. \tag{32}$$

To find the ground state, we follow two steps (see also Fig. 2)

- First, split each site i into i_L and i_R and place a spin 1/2 on each of the 2L sites of the chain and build a Valence bond solid (VBS)

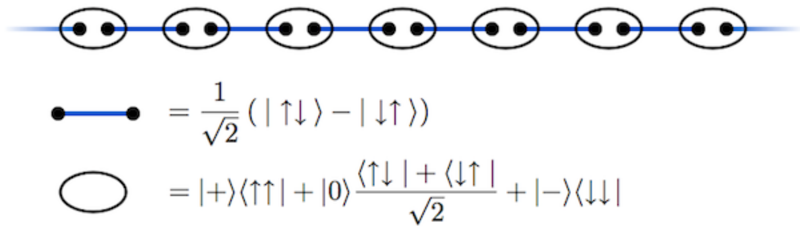


Figure 2: Schematic representation of the AKLT ground state (*picture stolen from wikipedia*)

- Of course, we will later have to project onto the triplet subspace of the 4D Hilbert space at each pair of i_R, i_L sites.
- We introduce a singlet on each pair of i_R and $(i+1)_L$

$$|\text{singlet}_{i_R,(i+1)_L}) = \underbrace{\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}}_{R_{\sigma_{i,R},\sigma_{i+1,L}}} |\sigma_{i,R})|\sigma_{i+1,L}). \tag{33}$$

  Here, we use round brackets do denote the fictitious Hilbert space with two spin-1/2 particles per site.
- Repeating this over the entire chain, we obtain the VBS wave function

$$|\text{VBS}) = |\sigma_{1,L}) \left(\prod_{i=1}^{L-1} R_{\sigma_{i,R},\sigma_{i+1,L}} |\sigma_{1,R}, \sigma_{2,L} \ldots, \sigma_{L,L}) \right) |\sigma_{L,R}). \tag{34}$$

  Note that, for open boundary conditions, this wave function leaves the spins (1,L) and (L,R) uncontracted – they are free. For closed boundary conditions, there is one more R matrix

- For a pair of adjacent sites, there are thus four spins. Since we imposed that the central two spins form a singlet, the overall pair of sites can only be in spin-1 or spin-0 state and thus is projected out by application of the Hamiltonian.
- So all that's left is to project back into the physical Hilbert space using

$$P_i = M^{s_i}_{\sigma_{i,L},\sigma_{i,R}} |1, s_i\rangle\, (\sigma_{i,L}|(\sigma_{i,R}|, \tag{35}$$

  where $s_i \in \{-1, 0, 1\}$ and

$$M^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{36}$$

$$M^0 = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}, \tag{37}$$

$$M^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{38}$$

– Thus, the physical (= after projection) state obtained from the VBS state is

$$\begin{aligned} |GS\rangle &= \left(\prod_{i=1}^{L-1} M^{s_i}_{\sigma_{i,L},\sigma_{i,R}} R_{\sigma_{i,R},\sigma_{i+1,L}}\right) M^{s_L}_{\sigma_{L,L},\sigma_{L,R}} |s_1,\ldots,s_L\rangle \\ &= (M^{s_1} R M^{s_2} R \ldots R M^{s_L})_{\sigma_{1,L},\sigma_{L,R}} |s_1,\ldots,s_L\rangle \end{aligned} \tag{39}$$

- Comments about the ground state
  - This state is annihilated by the application of the Hamiltonian by construction because it can contain at most a spin-1 state at a given rung
  - It is the paradigmatic example not only for the state of an SPT, but also for a matrix-product state
  - In the case of open boundary conditions, we have the dangling spin-1/2 at the boundaries, see Fig. 2.
  - In the case of closed boundary conditions, the ground state is fully gapped and

$$|GS\rangle = \mathrm{tr}\left[M^{s_1} R M^{s_2} R \ldots R M^{s_L} R\right] |s_1,\ldots,s_L\rangle . \tag{40}$$

  - Clearly, the $|GS\rangle$ which we constructed is a singlet. It is also possible to show that the ground state has to be a singlet – this is not surprising in view of the overall antiferromagnetic interactions. Specifically, the arguments go by contradiction
    * First, note that the open chain has two spin 1/2 edge states which form a singlet and a triplet.
    * Now imagine that the ground state of the bulk was a spin 1 state with $m_z = 1$. Then, one could in principle combine it with the edge states and integrate out the bulk to end up with edge states with $m_z = 2$.
    * but we also showed that the edge states are spin $0, 1$, so the bulk can not be spin 1.

Part II

Topological order

In the previous part of this lecture course we studied symmetry protected topological order concentrating on 1D systems.

In higher dimension, there are states which are even 'more topological' inasmuch as they do not have the symmetry requirement.

Defining characteristics of Topological order

- The ground state(s) is/are separated from the higher states by a gap.
- massive entanglement.
- Ground state degeneracy on a closed manifold with "holes"
- Apparance of excitations with non-trivial exchange statistics ("anyons")

The last three points are actually interconnected.

It is important that again, there is no local order parameter.

We will illustrate this type order by the simplest possible model displaying topological order.

6 The toric code

The toric code is

- a topological quantum error correction code
- a quantum spin liquid
- the integrable limit of a $\mathbb{Z}_2$ lattice gauge theory

at the same time.

It is defined by the Hamiltonian

$$H = -K\sum_{\square} B_{\square} - g\sum_{\mathbf{b}} \sigma^x_{\mathbf{b}} - J\sum_{\mathbf{b}} \sigma^z_{\mathbf{b}} - h\sum_{\mathbf{r}} Q_{\mathbf{r}}. \tag{41}$$

Comments

- This model is defined by spin 1/2 operators on the bonds $\mathbf{b}$ of the square lattice.
- The flux term is $B_{\square} = \prod_{\mathbf{b}\in\partial\square} \sigma^z_{\mathbf{b}}$ and the star operator is $Q_{\mathbf{r}} = \prod_{\mathbf{b} \text{ adjacent to } \mathbf{r}} \sigma^x_{\mathbf{b}}$.

6.1 Exact solution at $g = J = 0$

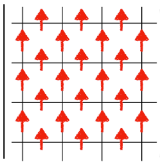
In the limit $J = g = 0$ the problem is exactly soluble.

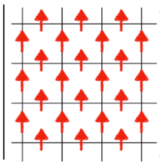
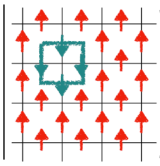
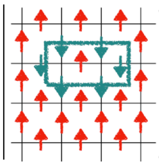
- Overall, the Hilbert space dimension is 2×2 per unit cell.
- All $Q_{\mathbf{r}}$ and $B_{\square}$ commute and thus also commute with the Hamiltonian: They are integrals of motion.
- Moreover $Q^2_{\mathbf{r}} = 1$ and $B^2_{\square} = 1$, so that the eigenvalues of either operator are ± 1 and the Hilberspace dimension spanned by eigenstates of these operaors is again 2×2 per unit cell.
- Thus the number of integrals of motion equals the number of degrees of freedom and since we can diagonalize all flux and star terms at the same time any state can be written as
$$|\psi\rangle = |\{Q_{\mathbf{r}}\}, \{B_{\square}\}\rangle\,. \tag{42}$$

6.1.1 Groundstate

We now focus on the case $K > 0$ and $h > 0$. In the following we construct the ground state explicitly. This is best done pictorially

$$|GS_0\rangle = |\{Q_r = 1\}, \{B_\square = 1\}\rangle$$

$$= \prod_{\mathbf{r}} \frac{1 + Q_{\mathbf{r}}}{2} \Big|$$  $$\Big\rangle$$

$$= \Big|$$  $$\Big\rangle + \Big|$$  $$\Big\rangle + \Big|$$  $$\Big\rangle \ldots$$

Comments

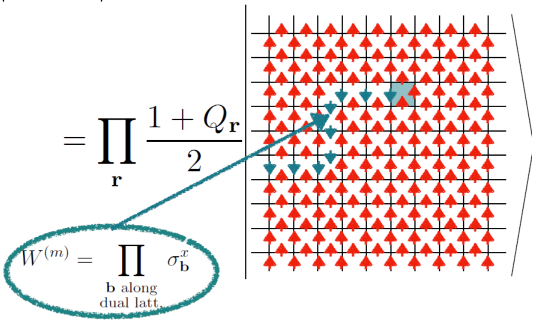
- Indeed, this term has all $Q_{\mathbf{r}} = 1$ and alld $B_\square = 1$:
  - The product $\prod_{\mathbf{r}} \frac{1+Q_{\mathbf{r}}}{2}$ clearly projects onto states with $Q_{\mathbf{r}} = 1$
  - Since all $B_\square$ commute with all $Q_{\mathbf{r}}$, we can commute all $B_\square$ accross the projector at no cost. Then, they act on a state which clearly has zero flux through all plaquettes.
- In the third line, we expand the product over projectors
  - Note that each $Q_{\mathbf{r}}$ acting on the "all-up" state flips all four spins adjacent to a given site
  - Therefore, the expansion of the product of projectors leads to a superposition of states characterized by closed loops of flipped spins.
- Therefore, the Toric code ground state is
  - A superposition state of macroscopically many quantum states ($\Rightarrow$ highly entangled!)
  - A simple example of a "string-net-condensate"

6.1.2 Excitations

The simplest excitation is to flip a single plaquette or star. For example, if we flip the plaquette $\square_0$, the state is pictorially represented as

$|m : \square_0\rangle =$

$$= \prod_{\mathbf{r}} \frac{1+Q_{\mathbf{r}}}{2}$$



Comments:

- Such an excitation is called an m particle, because it corresponds to one flux. (By analogy, when we flip the star, it would be called e particle)
- Clearly, a single m has energy $2K$ above the ground state. (A single e has energy $2h$)
- We introduced the string operator $W^{(m)} = \prod_{\mathbf{b}\text{ along dual latt.}} \sigma^x_{\mathbf{b}}$ along a dual lattice. By analogy there is $W^{(e)} = \prod_{\mathbf{b}\text{ along latt.}} \sigma^z_{\mathbf{b}}$
- On a closed manifold (e.g. a torus) each string operator has to have zero or two ends (so an m never appears alone)
- We will see later that m and e are actually anyons, but there is one more anyon in the theory (see below).

Fusion rules:

- It's pretty obvious that $m \times m = 1$ and $e \times e = 1$, and obviously $1 \times a = a$ for any anyon a.
- However, what is $e \times m$? Clearly it's neither e, m or 1, so it's something new and we call it ϵ. Then, $\epsilon \times m = e, \epsilon \times e = m$.

6.1.3 Exchange statistice of excitations

Let's first remember the basics about the wave functions for identical particles:

- For identical particles, an exchange of positions may at most change the wave function of the many body sytem by a phase $|a_1 : \mathbf{r}_1; a_2 : \mathbf{r}_2\rangle = e^{i\theta} |a_1 : \mathbf{r}_2; a_2 : \mathbf{r}_1\rangle$. *(In some more exotic cases, the phase can also be promoted to a unitary matrix)*

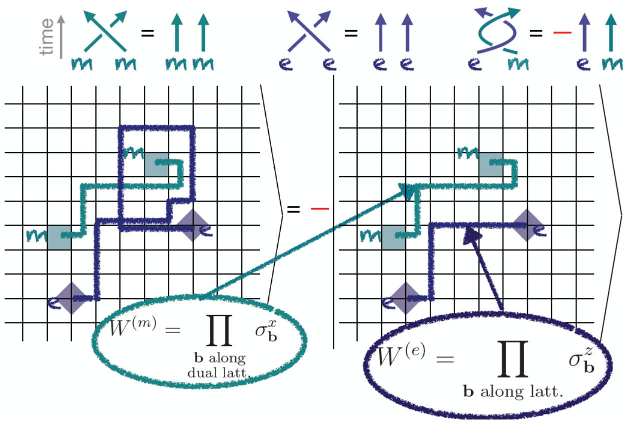


Figure 3: Pictorial representation of exchange statistics of anyons in the toric code model.

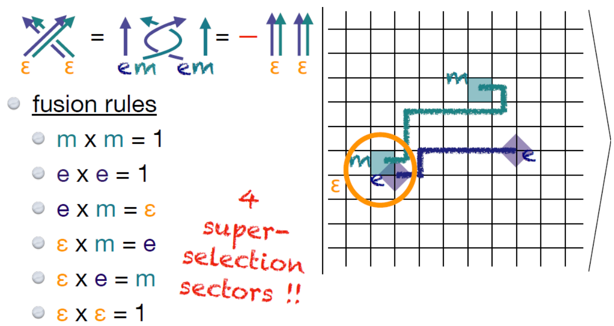


Figure 4: Fusion rules and exchange statistics of ϵ.

- Double exchange, which is equivalent to moving a_1 on a loop around a_2 , thus leads to a phase $e^{i2\theta}$.
- In 3D and for particle like excitations, any loop may be contracted to a point. Thus $e^{i2\theta} = 1$ or $\theta = 0, \pi$ ("boson" or "fermion").
- In 2D, loops can not be contracted to a single point and thus anyons with arbitrary statistical angle θ can exist.

Let's now see what happens to the anyons in our system

- Clearly, moving an m around and m does not change the phase of the wave function. Thus m behaves like a boson to itself. The same is true for e.
- However, when we move an e around an m, the wave function acquires one extra $-$ sign, because the strings interact an odd number of times. Thus e and m are "mutual semions', see fig. 3
- As a corrolary, ϵ pariticles have fermionic statistics (best seen pictorially, Fig. 4).

So in total, from a problem which only included spins ("qubits"), we were able to obtain excitations which are fermions!!!

6.1.4 Topological entanglement entropy

In this section, we quantify the notion of topological entanglement.

- *Definition: Entanglement entropy.*
  - First, take a bipartition of the system into subparts A and B.
  - Reduced density matrix: $\rho_A = \mathrm{Tr}_B \left|GS\right\rangle \left\langle GS\right|$.
  - (von-Neumann) entanglement entropy $S = -\mathrm{tr}\left[\rho_A \ln \rho_A\right]$
- Universal, topological entanglement entropy γ: $S = \#L - \gamma$, where for Abelian anyons $\gamma = \ln(\#\text{superselection sectors})/2$. The first term reflects "area-law" scaling and is non-universal. For the toric code $\gamma = \ln(2)$.
- Handwavy explanation for the $\gamma = \ln(2)$:
  - Loops residing only in A or B won't contribute to S, only those which cross the boundary.

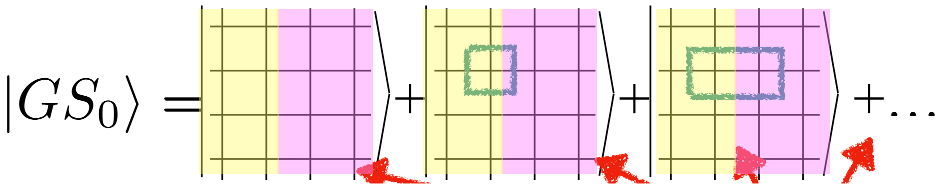


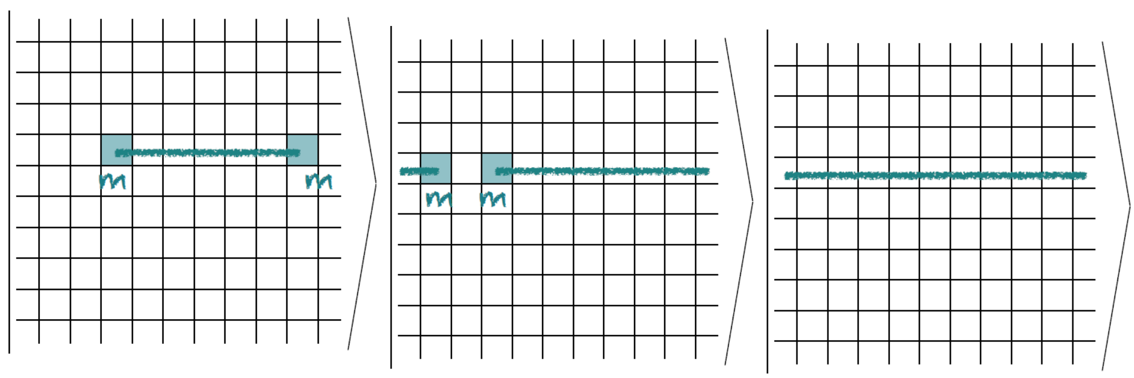
Figure 5: Illustration of the topological entanglement entropy.

- Since all loops are closed, the number of σ_x-strings crossing the bipartition is even (i.e. the number of spin flips is even)
- This single constraint reduces the possible configurations of spins at the boundary form 2^L to 2^{L-1} leading to an entropy $\ln 2^{L-1} = (L-1)\ln(2)$. (see Fig. 5)

6.1.5 Groundstate degeneracy on the torus

The appearance of anyons also implies a non-trivial ground state degeneracy on a torus.

To see this, we successively build the magnetic holonomy $X_1 = \prod_{\mathbf{b}\in\gamma_{\text{horizontal}}} \sigma^x_{\mathbf{b}}$.



Question: Is $X_1 |GS\rangle = |GS\rangle$ (modulo a phase)?

Answer: To find an answer, let's build an electric holonomy $Z_1 \prod_{\mathbf{b}\in\gamma_{\text{vertical}}}$ along vertical bonds. Clearly Z_1 anticommutes with X_1, and thus they form a 2D Hilbert space.

Similarly, one can construct yet another pair of holonomies Z_2 and X_2, which forms yet another 2D hilbert space (see Fig. 6). So in total the ground state degeneracy is 4.

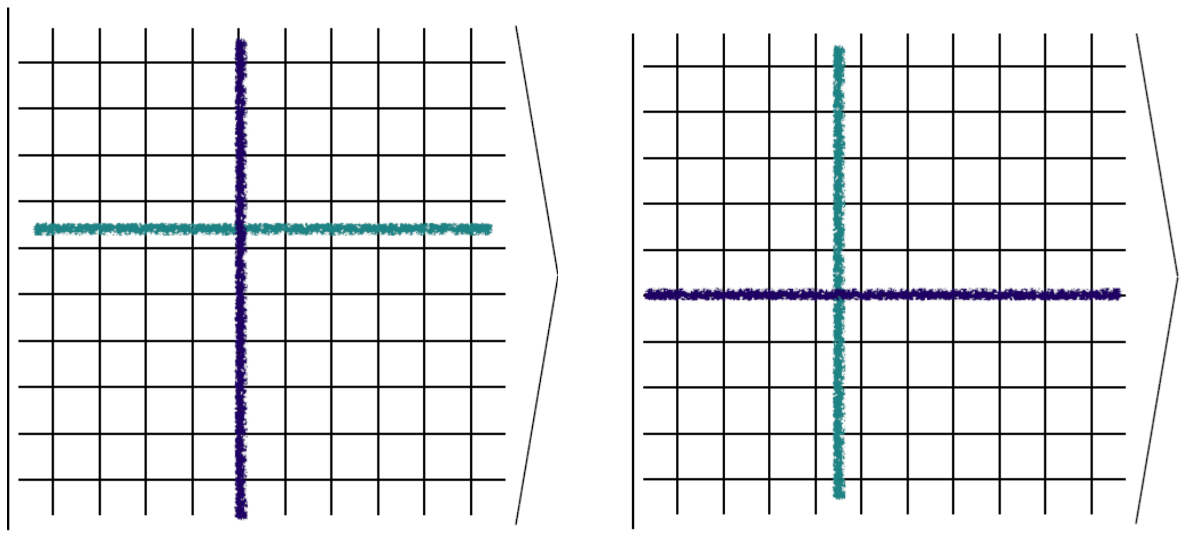


Figure 6: Left: Illustration of the two holonomies X_1, Z_1, Right: The same for X_2, Z_2.

7 Perturbations: $g, J \neq 0$

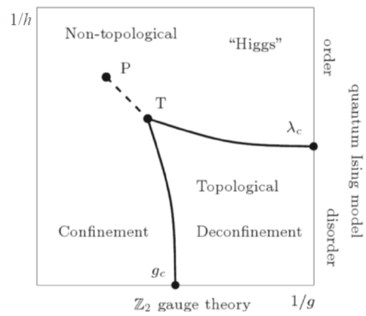
We briefly summarize the properties of the full Hamiltonian, Eq. (41), away from the integrable Toric Code limit $J = g = 0$. Importantly, due to the energy gap, the non-trivial topological ground state is protected towards the inclusion of finite g, J.

- Ground state splitting: $\Delta E \sim \exp(- L/\xi)$
  - robustly encode quantum information

    => "Toric Code"
  - $\langle Z_{1,2} \rangle \sim e^{-L/\xi}$: perimeter law of Wilson-loops

    <=> "Deconfinement"
- Phase Diagram

  (for J = 1/h, and K = 1/g)



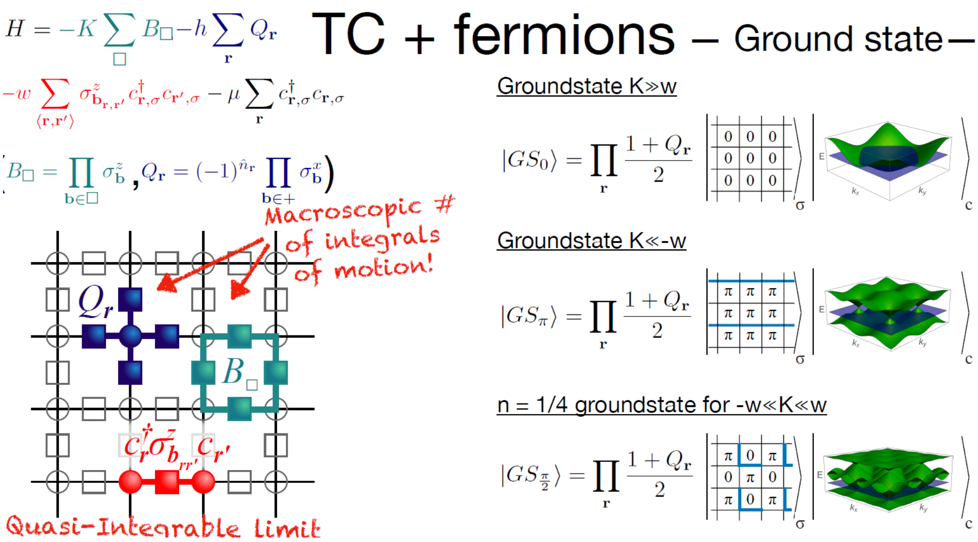
[Fradkin & Shenker (1979)]

8 Luttinger's theorem, Fermi surface reconstruction and orthogonal metals

In this section we discuss the way topological order allows to circumvent Luttinger's theorem and explicitly illustrate this by using an examplary, soluble toy model.

8.1 Fermions Coupled to the Toric Code

The model under consideration and the groundstates (which can be constructed in the same way as what we did for the toric code) are summarized in the following slide



Comments

- It's important to emphasize that fermions can be chosen spinless and that σ matrices on the bonds are not spin!
- Note the different definition of $Q_{\mathbf{r}}$ which now also includes fermion parity and the σ^z dependent hopping.
- This model can not be solved exactly (i.e. not all states are known exactly (the number of integrals of motions does not coincide with the the number of degrees of freedom.
- In the limit $|K| \gg w$, the groundstates are however exactly known. In particular, for $K < 0$, we construct the effective dispersion in Fig. 7

Here, we solve the π flux model in the gauge where $\langle \sigma^z_{\boldsymbol{b}} \rangle = (-1)^{b_x}$, i.e. it is negative on every other vertical column but positive everywhere else (see Fig. 7). We choose a two atom unit cell of dimers along the x direction and Fourier transform

$$c_{\boldsymbol{x},1} = \int_{\text{small BZ}} (dk) e^{i\boldsymbol{k}\boldsymbol{r}} c_{\boldsymbol{k},1} \tag{B1}$$

$$c_{\boldsymbol{x},2} = \int_{\text{small BZ}} (dk) e^{i\boldsymbol{k}\boldsymbol{r}+ik_x x} c_{\boldsymbol{k},2} \tag{B2}$$

(Note that 1, 2 labels do not correspond to sublattice labels A, B). The momentum space Hamiltonian is

$$H = -2w \int_{\text{small BZ}} (dk) c^\dagger_{\boldsymbol{k}} [-\cos(k_y)\gamma_z + \cos(k_x)\gamma_x] c_{\boldsymbol{k}} \tag{B3}$$

[the small Brillouin zone (BZ) is $\boldsymbol{k} \in (-\pi/2, \pi/2) \times (-\pi, \pi)$] which implies a dispersion

$$\epsilon^{(\pm)}_\pi(\boldsymbol{k}) = \pm 2w\sqrt{\cos(k_x)^2 + \cos(k_y)^2}. \tag{B4}$$

Dirac nodes occur at $|k_x| = |k_y| = \pi/2$.

Figure 7: Solution of the fermionic hopping problem in the case of the π-flux phase

- For $|K| \lesssim w$, one may study some trial ground states.

8.2 Discussion of different types of topological order

- The states at $K \gg w$ and $K \ll w$ do not differ by any symmetry breaking pattern, but by means of their topological order.
- A way to categorize states with different topological order are projective representation of symmetries
  - Normally, symmetries are represented in the Hilbert space by some unitary transformations with the property $R(g)R(h) = R(gh)$ for any two group elements g, h. A state is "symmetric" w.r.t to this operation, if it is left invariant by the transformation.
  - Projective representations of symmetries are characterized by $R(g)R(h) = e^{i\theta(g,h)}R(gh)$
  - They appear when unitary transformations leave the system invariant modulo a gauge transformation.

We illustrate this concept studying translational symmetries.

- usually (and in the case $K \gg w$) they satisfy $R(T_x^{-1}) \circ R(T_y^{-1}) \circ R(T_x) \circ R(T_y) = 1$.

- Let's carefully do this operation for the trial state for $K \ll (-w)$ *(The projector just ensures the sum over all possible equivalent gauge configurations – it could also be enforced as a "Gauss' law")*
  * Translations by $\pm\hat{e}_y$: Supplemented by gauge transformation $(-1)^x$ (applied to c variables) to keep the state invariant.
  * Translations by $\pm\hat{e}_x$: leaves state invariant.
  * Thus we find a non-trivial projective representation of the symmetry.

$$R(T_x^{-1}) \circ R(T_y^{-1}) \circ R(T_x) \circ R(T_y) = \hat{T}_x^{-1}\hat{T}_y^{-1} \underbrace{(-1)^x \hat{T}_x (-1)^x}_{=-1} \hat{T}_y = -1 \quad (43)$$

(This representation of translations is also called "magnetic translation operators")

8.3 Luttinger-Oshikawa Theorem

In this section, we first "derive" the Luttinger-Oshikawa theorem and then, following arguments by Paramekanti and Vishwanath, we explain a loophole in the theorem.

Luttinger-Oshikawa theorem *(2D and spinless fermions, for simplicity)*

Consider a square lattice Hamiltonian (lattice spacing a), e.g. the Hubbard model for a 2D system of size $L_x \times L_y$ and total particle number $N = nL_xL_y$, and assume a Fermi liquid state (i.e. all symmetries are preserved and there is a Fermi surface where all charged low-energy excitations are fermionic quasi-particles close to the Fermi surface). The volume of the Fermi surface is then fixed by the density, independently of the interactions

$$\frac{V_{\text{FS}}}{4\pi^2} = \frac{N}{L_xL_y} \mod \frac{1}{a^2} \quad (44)$$

The "mod$(1/a^2)$" reflects filled bands.

"Proof." The classic proof for continuum systems involves a careful consideration of Ward-identities related to particle number conservation and Galilean invariance. It goes back to Luttinger and can be found in the textbook by Abrikosov, Gor'kov, Dzyaloshinskii.

Here we present the essence of the proof devoloped by Oshikawa in the context of Kondo lattice systems. It is a topological flux insertion argument and the Luttinger theorem follows from a momentum balance argument.

Consider a cylinder geometry and insert flux. After the adiabatic insertion of a flux 2π through the hole of the cylinder the many-body Hamiltonian is gauge equivalent to the

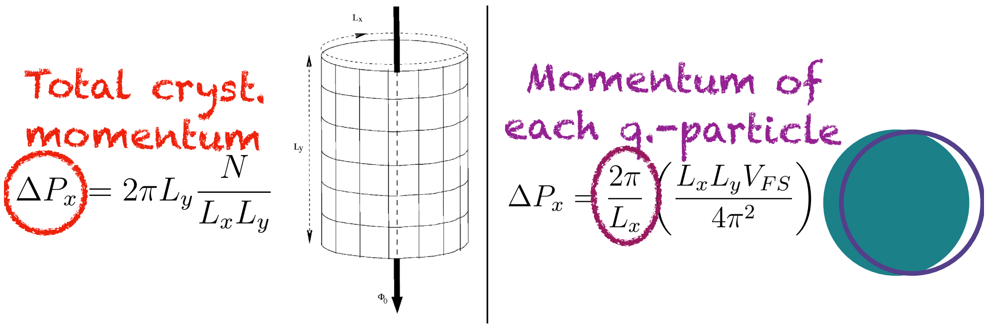


Figure 8: Momentum balance following Luttinger Oshikawa arguments. On the right, the purple circle corresponds to $n(\mathbf{p} - \delta p \hat{e}_x)$, while the filled circle is $n(\mathbf{p})$

Hamiltonian prior to the insertion of the flux. Thus all eigenstates are the same. Yet a momentum has been transferred which we count in two different ways.

i By simple electrodynamics, the cylinder now spins due to the electromotive force and the total (angular) momentum of the system is

$$\Delta P_x = 2\pi L_y n \tag{45}$$

(where n is the average particle number per site).

ii In a Fermi liquid, the momentum can only be carried by quasi-particles near the Fermi surface. As their quantum numbers are the same as those of free electrons, we know that they acquire an additional momentum $\delta p_x = 2\pi/L_x$. The overall momentum difference can thus be calculated in a different way, namely

$$\begin{aligned}\Delta \vec{P} &= L_x L_y \int \frac{d^2p}{(2\pi)^2}[n(\mathbf{p} - \delta\mathbf{p}) - n(\mathbf{p})]\mathbf{p} \\ &= -L_x L_y \delta p_x \int \frac{d^2p}{(2\pi)^2} \partial_{p_x} n(\mathbf{p})\mathbf{p} \\ &\overset{\text{partial Int.}}{=} \frac{L_y}{2\pi} V_{\text{FS}} \hat{e}_x \end{aligned} \tag{46}$$

Equating Eq. (45) and (46) and keeping in mind that momenta are only conserverd modulo reciprocal lattice vectors, we end up at Eq. (44), see Fig. 8 *(A more thorough treament will include the discreteness of the lattice and flux insertions in both x and y directions, but is conceptually the same.)*

"Loophole" in Luttinger's theorem

We now find a "loophole" to Luttinger's theorem allowing for a smaller Fermi surface even if all symmetries are preserved. We illustrate this for the state of our fermionic Toric

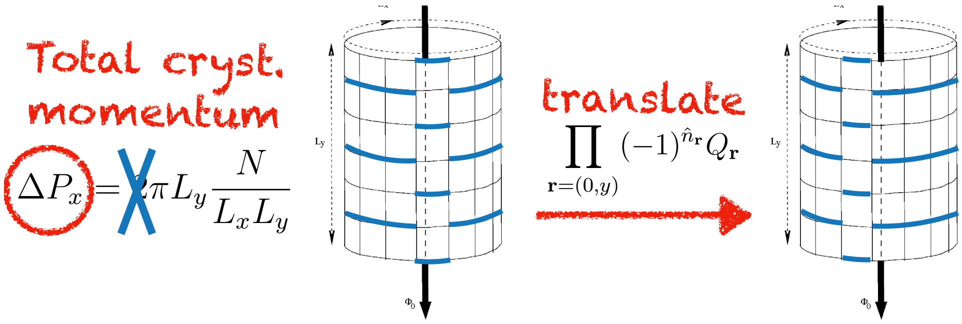


Figure 9: Momentum balance following Luttinger Oshikawa arguments for the case $K \ll -w$.

code at $K \ll -w$ and at half filling (i.e. we demonstrate that momentum balance can be achieved even when $V_{FS} = 0$)

We adiabatically insert a flux π, by Faraday this leads to an overall momentum of $\Delta P_x = \pi L_y n$.

Of course, insertion of only flux π does not map the system back to itself - but we can absorb this flux π in the $\mathbb{Z}_2$ gauge sector, see Fig. 9. This is equivalent to including a magnetic holonomy along the dashed line

$$H \rightarrow W^{(m)} H W^{(m)} \tag{47}$$

and thus $|GS_\pi\rangle \rightarrow W^{(m)} |GS_\pi\rangle$. We now demonstrate that, indeed, the momentum difference between these to states $\Delta P_x = \pi L_y n$, even in the absence of a Fermi surface. (The momentum is carried by the gauge sector instead).

Indeed, translation by one lattice constant is equivalent to multiplication with a string $(-1)^{\hat{n}_r} Q_\mathbf{r}$ along the line, and thus

$$\prod_{\mathbf{r}\text{with } x=0} (-1)^{\hat{n}_r} Q_\mathbf{r} |GS_\pi\rangle = \prod_{\mathbf{r}\text{with } x=0} (-1)^{\langle \hat{n}_r \rangle} |GS_\pi\rangle = e^{i\pi n L_y} |GS_\pi\rangle . \tag{48}$$

Here, translational symmetry of the fermionic ground state was used.

Thus, the momentum balance is fulfilled for this state, even without invoking the existence of a surface of low-lying fermionic quasiparticles.

8.4 Fermionic correlators and orthogonal metal

Let's consider the case $K \gg w$ and calculate the fermionic Green's functions ($\tau > 0$)

$$\begin{aligned} G_{\mathbf{x},\mathbf{x}'}(\tau) &= -\langle GS|c_{\mathbf{x}}(\tau)c^{\dagger}_{\mathbf{x}'}(0)|GS\rangle \\ &= e^{-2h\tau}\delta_{\mathbf{x},\mathbf{x}'}G^{\mathrm{FS}}_{\mathbf{x},\mathbf{x}}(\tau). \end{aligned} \tag{49}$$

Here we used $H = H_c + H_{\mathrm{TC}}$ corresponding to the fermionic and toric Code Hamiltonians respectively. Then $c(\tau) = e^{H\tau}ce^{-H\tau} = e^{-2hQ_{\mathbf{r}}}e^{H_c\tau}ce^{-H_c\tau}$ and that $Q_{\mathbf{r}}c_{\mathbf{r}}|GS\rangle = -c_{\mathbf{r}}|GS\rangle$, so that the overlap vanishes when there is an odd number of fermion operator insertions on a given cite. The superscript "FS" stands for "Fermi surface" and is the standard fermionic Green's function on the square lattice for this problem.

In contrast, non of these aspects affect the polarization bubble

$$\begin{aligned} \Pi_{\mathbf{x},\mathbf{x}'}(\tau) &= -\langle GS|[c^{\dagger}c]_{\mathbf{x}}(\tau)[c^{\dagger}c]^{\dagger}_{\mathbf{x}'}(0)|GS\rangle \\ &= \Pi^{\mathrm{FS}}_{\mathbf{x},\mathbf{x}}(\tau). \end{aligned} \tag{50}$$

Comments

- Thus, the single particle fermion Green's function is gapped, and ultrashort ranged. This shows up in STM.
- However, density correlations, and current etc. behave as for a metal
- Such a state of matter has been called "orthogonal metal" (Nandkishore et al PRB 2012)

Exercise (based on Abanov lecture notes)

1. Quantum particle on a ring threaded by flux

Consider a particle on a ring characterized by ϕ (the angle corresponding to the particle's position) and A (a constant characterizing the flux $\Phi = 2\pi A$ threading the ring)

$$H = \frac{(-i\partial_\phi - A)^2}{2M}. \tag{51}$$

1A. Spectrum

Diagonalize Eq. (51). What is the spectrum, keeping in mind the compact nature of the base manifold? What is the ground state degeneracy as a function of A? Write down the partition sum without evaluating the sum.

1B. Path integral description

Demonstrate that in Feynman path integral description, the Hamiltonian Eq. (51) maps to a Euclidean action $S = S_{\text{kin}} + S_\theta$ with

$$S_{\text{kin}} = \int_0^\beta d\tau \frac{M\dot{\phi}^2}{2}, \quad S_\theta = -i\frac{\theta}{2\pi}\int_0^\beta d\tau\dot{\phi}. \tag{52}$$

Determine θ.

1C. Semiclassical equation of motion

Derive the semiclassical equation of motion and particularly focus on the way θ enters.

1D. Topological nature of theta term

Show that $S_\theta = -i\theta Q$, where $Q \in \mathbb{Z}$, and explain the meaning of Q and the parametrization ($\tilde{\phi}(\beta) = \tilde{\phi}(0)$)

$$\phi(\tau) = \tilde{\phi}(\tau) + 2\pi Q\tau/\beta. \tag{53}$$

1E. Partition function

Using the parametrization Eq. (53), demonstrate that the path integral takes the form

$$\mathcal{Z}_\theta = \sum_{Q=-\infty}^{\infty} \int \mathcal{D}\tilde{\phi} e^{-S_{\text{kin}}+i\theta Q} \tag{54}$$

Motivate the wording "θ-angle". What is θ an angle of? Calculate the relative partition sum $\mathcal{Z}_\theta/\mathcal{Z}_{\theta=0}$ and compare to the result obtained in subexercise A.

Hint: Poisson resummation formula $\sum_{m=-\infty}^{\infty} h(m) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\varphi e^{-2\pi i n\varphi} h(\varphi)$

Solution to Exercise

1. Quantum particle on a ring threaded by flux

1A. Spectrum

It is convenient to us the angular momentum quantum number $m \in \mathbb{Z}$ to find the spectrum $E_m = (m-A)^2/2M$. When $A \in \mathbb{Z}+1/2$, the ground state is doubly degenerate, otherwise it is non-degenerate. The partition sum is

$$\mathcal{Z} = \sum_m e^{-E_m/T} \tag{55}$$

1B. Path integral description

We find $\theta = 2\pi A$, i.e. the flux through the ring.

1C and D. Semiclassical equation of motion and topological nature of the theta term.

The semiclassical equation of motion is independent of θ, $\ddot{\phi} = 0$ and the parametrization of $\phi(\tau) = \tilde{\phi}(\tau) + 2\pi Q\tau/\beta$ splits the winding Q of the phase from (which enters the theta term a quantized topological charge) a topologically trivial smooth field configurations $\tilde{\phi}(\tau)$.

1E. Partition function The partition function is calculated by splitting the sum over distinct topological sectors from the integral over smooth field configurations. Thus the angle θ characterizes the quantum phase between the contributions of classes of trajectories with adjacent winding numbers.

To calculate the partition function, it is convenient to expand $\tilde{\phi}(\tau) = \sum_l \phi_l e^{i2\pi l\tau/\beta}$, where $\phi_l = \phi^*_{-l}$)

$$\mathcal{Z}_\theta = \sum_{Q=-\infty}^{\infty} e^{i\theta Q - 2\pi^2 MQ^2/\beta} \int \mathcal{D}[\phi_l, \phi_l^*] \exp[-\sum_{l=-\infty}^{\infty} 2\pi^2 M l^2 |\phi_l|^2/\beta]. \tag{56}$$

We next use the Poisson summation formula

$$\begin{aligned} \sum_{Q=-\infty}^{\infty} e^{i\theta Q - 2\pi^2 MQ^2/\beta} &= \sum_{n=-\infty}^{\infty} \int d\varphi e^{i(\theta+2\pi n)\varphi - 2\pi^2 M\varphi^2/\beta} \\ &\propto \sum_{n=-\infty}^{\infty} e^{-\frac{(\theta+2\pi n)^2}{8\pi^2 MT}}, \end{aligned} \tag{57}$$

and thus we find $\mathcal{Z}_\theta/\mathcal{Z}_0$ consistent with Eq. (55).