Field theories of disordered condensed-matter systems

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Contents

Literature (Textbooks).

- [Altland and Simons(2010)]: *Chapters 6.5, 8.5, 9.3.7. Replica Approach*
- [Efetov(1999)]: *Textbook exclusively devoted to the topic. Supersymmetry Approach*
- [Kamenev(2011)]: *Chapters 11-14. Keldysh Technique*
- [Mirlin(2000)]: *Lecture Notes on Supersymmetry*
- [Wegner(2016)]: *Chaps 4, 21-23. Mostly Supersymmetry Approach.*

For the part on disorder and interactions, three standard reviews are on the market [Finkelstein(1990), Belitz and Kirkpatrick(1994), Finkel'stein(2010)].

For the SYK part, there are reviews by [Trunin(2021), Rosenhaus(2019)] as well as lecture notes by Sachdev at `https://qpt.physics.harvard.edu/talks/jerusalem19a.pdfA`.

Part I

Single particle quantum mechanics and disorder

1 Fundamentals of disordered systems

1.1 Perturbation theory and Diagrams

The quantum mechanical single particle problem of motion in an impure environment is described by a Hamiltonian which includes the superposition of many impurity potentials.

$$\hat{H} = \hat{H}_{\text{kin}} + V(\hat{\mathbf{r}}); \quad V(\hat{\mathbf{r}}) = \sum_i v(\hat{\mathbf{r}} - \mathbf{R}_i). \tag{1}$$

For simplicity, we here consider $v(\mathbf{r} - \mathbf{R}) = V_0 \delta(\mathbf{r} - \mathbf{R})$. A diagrammatic theory follows from the expansion of

$$\hat{G}_R(E) = [E + i\eta - \hat{H}]^{-1} = \sum_{n=0}^{\infty} \hat{G}_0(E)[\hat{V}\hat{G}_{R,0}(E)]^n, \quad \hat{G}_{R,0}(E) = [E + i\eta - \hat{H}_{\text{kin}}]^{-1}. \tag{2}$$

Typically, an impurity is represented as cross and the potential as a dashed line, see Fig. 1.

Whenever one is interested in macroscopic response functions, it is plausible to treat the positions $\mathbf{R}$ of impurities statistically and average

$$\langle \dots \rangle_{\text{dis.}} = \int \prod_i \frac{d^d R_i}{L^d} \dots . \tag{3}$$

The impurity problem is thus equivalent to a problem made up of a random potential with a given probability distribution. In the limit when impurities are infinitely weak but infinitely dense

$$n_{imp} V_0^2 = const.; \quad n_{imp}^{-1} \to 0, V_0 \to 0 \tag{4}$$

this distribution is Gaussian and white noise, see Fig. 1 b, c.

$$\langle V(\mathbf{r}) V(\mathbf{r}') \rangle_{\text{dis.}} = n_{\text{imp}} V_0^2 \delta(\mathbf{r} - \mathbf{r}') \tag{5a}$$

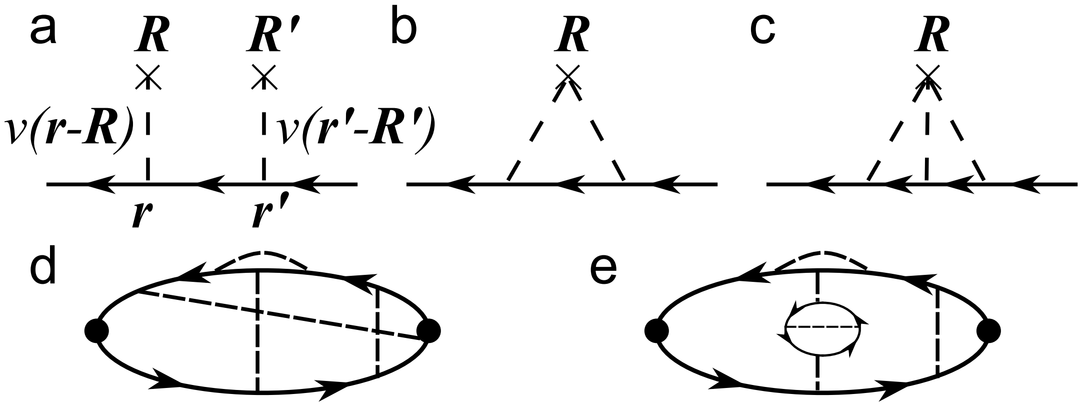


Figure 1: Diagramatic representation of corrections due to disorder. a) A cross represents an impurity at position $\mathbf{R}$, and the dashed line $v(\mathbf{r}-\mathbf{R})$. b) After average, the leading non-trivial diagram contains two dashed lines connecting to the same impurity. It is thus $\mathcal{O}(n_{\text{imp}}V_0^2)$ c) This diagram is $\mathcal{O}(n_{\text{imp}}V_0^3)$ and negligible in the limit of dense weak impurities (Born limit). d) Disorder correction to a correlation function. e) A diagram which is reducible w.r.t. disorder lines and should be dropped.

which is equivalent to a disorder average

$$\langle \dots \rangle_{\text{dis}} = \int \mathcal{D}V \underbrace{\exp^{-\int d^dx \frac{V(\mathbf{x})^2}{2n_{\text{imp}}V_0^2}}}_{\mathcal{P}[V]} . \tag{5b}$$

We concentrate on the limit of Gaussian white noise disorder in the entire course. In this limit, the cross is typically dropped after disorder average and the dashed line simply represents the correlator Eq. 5a.

Thus, to evaluate the disorder average of an observable

- dress the clean diagram with disorder lines
- keep only diagrams which are irreducible w.r.t. to disorder lines, see Fig. 1 d,e.

The reason to drop reducible diagrams follows from the procedure of adding crosses to the Green's function and average afterwards.

1.2 Overview of methods to average $\ln(\mathcal{Z})$

Consider the polarization operator as a representative observable

$$\langle \Pi(\mathbf{x},\tau;\mathbf{x}',\tau')\rangle_{\text{dis.}} = -\langle \frac{\delta^2}{\delta\phi(\mathbf{x},\tau)\delta\phi(\mathbf{x}',\tau')} \ln(\mathcal{Z}[\phi])\Big|_{\phi=0} \rangle_{\text{dis.}}. \tag{6}$$

To obtain average observables - need average of $\ln(\mathcal{Z}[\phi])$ (or equivalently a method in which $\mathcal{Z}[\phi=0]=1$).

There are three known options

- Supersymmetry: Exploits that Bosonic and Fermionic determinants cancel

$$\mathcal{Z}_\zeta = \int \mathcal{D}\psi e^{-\int \bar{\psi}M\psi} = \det(M)^{-\zeta}, \quad \begin{cases} \zeta = 1, & \text{bosons,} \\ \zeta = -1, & \text{fermions,} \end{cases} \tag{7}$$

  and one actually introduces the full partition function as $\mathcal{Z}_{\text{SUSY}} = \mathcal{Z}_{\zeta=-1}\mathcal{Z}_{\zeta=+1}$

  + Mathematical rigor.
  + Leads to finite dimensional NLσM target manifolds (which can conveniently parametrized and exactly solved!)
  - Mathematical complexity.
  - Impractical for many-body physics.

- Keldysh formalism, which exploits that

$$\mathcal{Z}[\phi_{cl},\phi_q=0]=1. \tag{8}$$

  + Can also treat non-equilibrium
  + Can treat many-body interactions.
  - Symmetry group structure is mathematically less transparent.

- Replica trick

$$\ln\mathcal{Z} = \lim_{R\to 0}\frac{\mathcal{Z}^R - 1}{R}. \tag{9}$$

  + Simplest.
  + The omission of disorder reducible diagrams is obvious.
  + Can handle interactions.
  - Involves artificial and sometimes tricky analytical continuation from $R\in\mathbb{N}$ to $R\in\mathbb{R}$ and then limit $R\to 0$.

We stick to replicas throughout the lecture course.

2 Diffusion theory: "mean field" treatment of disordered systems

In this section we present the saddle point theory of disordered electron systems. Quantum fluctuations about this saddle point are included in the next chapter.

2.1 Derivation of the NLσM - orthogonal class AI

2.1.1 Disorder average of replicated partition function

We replicate the Matsubara partition function (in the first line we explicitly show summation symbols - in the remainder we will use Einstein convention)

$$
\begin{aligned}
S &= T\sum_{\epsilon_n}\sum_{\alpha=1}^{R}\sum_{\sigma=\uparrow,\downarrow}\int_{\mathbf{x}}\bar{\psi}_{\sigma,\alpha,n}(\mathbf{x})[-i\epsilon_n + H_{\text{kin}}(\hat{\mathbf{p}}) + V(\mathbf{x})]\psi_{\sigma,\alpha,n}(\mathbf{x}); &\text{(10a)}\\
\mathcal{Z}^R &= \left\langle \int \prod_{\alpha=1}^{R}\mathcal{D}[\bar{\psi}_{\sigma,\alpha},\psi_{\sigma,\alpha}]e^{-S}\right\rangle_{\text{dis.}}\\
&= \int \prod_{\alpha=1}^{R}\mathcal{D}[\bar{\psi}_{\sigma,\alpha},\psi_{\sigma,\alpha}]\Big\{\exp\left[-T\int_{\mathbf{x}}\bar{\psi}_{\sigma,\alpha,n}(\mathbf{x})(-i\epsilon_n + H_{\text{kin}}(\hat{\mathbf{p}}))\psi_{\sigma,\alpha,n}(\mathbf{x})\right]\\
&\quad\times\exp\left[T^2\frac{n_{\text{imp}}V_0^2}{2}\int_{\mathbf{x}}\bar{\psi}_{\sigma,\alpha,n}\psi_{\sigma,\alpha,n}\bar{\psi}_{\rho,\beta,m}\psi_{\rho,\beta,m}\right]\Big\} &\text{(10b)}
\end{aligned}
$$

We see that disorder introduces an effective attractive "interaction" which is local in space and completely non-local in time (remember that it is mediated by a field which is quenched).

The next step is to Hubbard-Stratonovich decouple this field in all possible channels:

- the density density (just undoing what we just did)
- the particle hole exchange channel
- the particle-particle (or Cooper channel).

Remarks:

- These channels are to be considered channels of small momentum transfer as compared to k_F - so we need to simultaneously decouple in all of them.
- The effect of the particle-particle channel is just to renormalize the chemical potential. It is therefore typically dropped.

2.1.2 Introducing Nambu space

Since we have to decouple in the Cooper channel it is wise to introduce Nambu spinors even before disorder average (in what follows $C = i\sigma_y\tau_x$)

$$\Phi_n = \sqrt{T}\begin{pmatrix} \psi_n \\ -i\sigma_y\bar{\psi}_n^T \end{pmatrix} \text{ and } \bar{\Phi} = (C\Phi)^T = \sqrt{T}(\bar{\psi}, \psi^T(-i\sigma_y)). \tag{11}$$

From now on summation over all internal indices (spin, Nambu, Matsubara, replica) is implied in a matrix multiplication.

Then, prior to disorder average

$$S = \frac{1}{2}\int_{\mathbf{x}} \bar{\Phi}(\mathbf{x})\begin{pmatrix} -i\hat{\epsilon}+\hat{H} & 0 \\ 0 & -i\hat{\epsilon}+\sigma_y\hat{H}^T\sigma_y \end{pmatrix}\Phi(\mathbf{x}) \tag{12a}$$

$$\overset{\text{average}}{\rightarrow} \quad S_0 - \frac{n_{\text{imp}}V_0^2}{8}\int_{\mathbf{x}} \bar{\Phi}_i\Phi_i\bar{\Phi}_j\Phi_j \tag{12b}$$

$$\overset{\text{HS in 2 channels}}{\rightarrow} \quad S_0 + \frac{1}{2}\int_{\mathbf{x}} \bar{\Phi}_i(-i)\frac{1}{2\tau}M_{ij}\Phi_j + \frac{1}{16\tau^2 n_{\text{imp}}V_0^2}\text{tr}\, M^2 \tag{12c}$$

We have used that

- We used a multiindex $i = (\sigma, \tau, \alpha, n)$
- For the simplest case, with slight abuse of notation, $H_{\text{kin}}(\hat{\mathbf{p}}) = -\nabla^2/2m - \mu$, $\hat{H}_{\text{kin}} = \sigma_y H_{\text{kin}}^T\sigma_y$
- $S_0 = -\frac{1}{2}\int_{\mathbf{p}} \bar{\Phi}_{\sigma\alpha n}(-\mathbf{p})\underbrace{[i\epsilon_n - H_{\text{kin}}(\mathbf{p})]}_{G_0^{-1}(\epsilon_n,\mathbf{p})}\Phi_{\sigma\alpha n}(\mathbf{p})$
- The matrix field M has the symmetry property

$$M = CM^TC^T. \tag{12d}$$

2.1.3 Saddle point equation

We seek a solution for spatially independent M, in which case

$$S \overset{\text{Integrate fermions}}{\rightarrow} \frac{1}{16\tau^2 n_{\text{imp}} V_0^2} \text{tr}\,[M^2] - \tfrac{1}{2} \text{Tr}\, \ln(\underbrace{\hat{G}_0^{-1} + \frac{i}{2\tau} M}_{\hat{G}_M^{-1}}) \tag{13}$$

Thus the saddle point equation is

$$\begin{aligned} 0 &= \frac{\delta S}{\delta M_{ji}} = \frac{M_{ij}}{8\tau^2 n_{\text{imp}} V_0^2} - \frac{i}{4\tau} \int_{\mathbf{p}} \underbrace{[G_M(\mathbf{p})]_{ij}}_{=\langle [\Phi\Phi^T C]_{ij} \rangle} \\ \Leftrightarrow \frac{M}{\tau} &= 2i n_{\text{imp}} V_0^2 \int_{\mathbf{p}} [i\hat{\epsilon} - H_{\text{kin}}(\mathbf{p}) + \frac{i}{2\tau} M]^{-1} \end{aligned} \tag{14a}$$

A low-energy solution (small $|\epsilon_n| \ll \mu$) to this equation is

$$M = \Lambda = \begin{pmatrix} \ddots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}_{\text{Matsubara}} \times \mathbf{1}_{\text{spin,Nambu,replica}}, \qquad \frac{1}{\tau} = 2\pi n_{\text{imp}} V_0^2 \tag{15}$$

Remarks

- Throughout the notes ν is the DOS per spin degree of freedom at the Fermi level.
- We'll discuss a set of other saddle point solutions in a second, Sec. 2.1.4 and some others in the part on level statistics, Sec. **??**

2.1.4 Symmetries, Goldstone manifold

We consider the symmetries of the UV action, Eq. (12a), disregarding the frequency part.

(This strategy is more obvious if we keep only two replicas and Wick rotate

$$i \begin{pmatrix} \epsilon_n & 0 \\ 0 & -\epsilon_n \end{pmatrix} \equiv i|\epsilon_n|\Lambda \to E \pm i\eta\Lambda|_{\eta\to 0}. \tag{16}$$

Clearly, in the $\eta \to 0$ limit, there frequency term just adds to the chemical potential and has no impact.)

We see that Eq. (12a) in the case conidered so far $\hat{H} \propto \mathbf{1}_\sigma$ (no structure in spin space) and $\hat{H} = \sigma_y \hat{H}^T \sigma_y$ has the continuous symmetry

$$\Phi(\mathbf{x}) \to T\Phi(\mathbf{x}), \quad \text{where } T^T \tau_x \sigma_y T = \tau_x \sigma_y, \text{ i.e. } T \in G = \mathrm{Sp}(2_\sigma \times 2_\tau \times 2M \times R), \tag{17}$$

Applying this rotation in the presence of the vacuum expectation of the field $M = \Lambda$, we find that

$$\Lambda \to Q \equiv T^{-1}\Lambda T \in G/H, \tag{18}$$

spans a continuous manifold of energetically equivalent mean field solutions.

The quotient space G/H of the Goldstone manifold is obvious when considering that

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}_{\text{Matsubara}} \in H \equiv \mathrm{Sp}(2_\sigma \times 2_\tau \times M \times R) \times \mathrm{Sp}(2_\sigma \times 2_\tau \times M \times R) \tag{19}$$

leaves the saddle point solution invariant.

2.2 Gradient expansion

We now consider slow variations of the mean field solutions inside the Goldstone manifold

$$Q(\mathbf{x}) = T^{-1}(\mathbf{x})\Lambda T(\mathbf{x}) \in G/H. \tag{20}$$

We return to the fermionic action prior to integrating fermions and include a vector potential ($[\hat{\mathbf{A}}]_{nn'} = \sum_m \mathbf{A}(\omega_m)\delta_{n-n',m}$, $\hat{\mathbf{A}}_{\tau_z} = \mathrm{diag}(\hat{\mathbf{A}}, -\hat{\mathbf{A}}^T)$) for future reference

$$\begin{aligned} S &= \int_{\mathbf{x}} \bar{\Phi}(\mathbf{x})[-i\hat{\epsilon} + \frac{(-i\nabla + \hat{\mathbf{A}}_{\tau_z})^2}{2m} - \mu - \frac{i}{2\tau}Q(\mathbf{x})]\Phi(\mathbf{x}) \\ &= \int_{\mathbf{x}} \bar{\Phi}'(\mathbf{x})[T(\mathbf{x})\left(-i\hat{\epsilon} + \frac{(-i\nabla + \hat{\mathbf{A}}_{\tau_z})^2}{2m}\right)T^{-1}(\mathbf{x}) - \mu - \frac{i}{2\tau}\Lambda] \underbrace{\Phi'(\mathbf{x})}_{=:T(\mathbf{x})\Phi(\mathbf{x})} \\ &= \int_{\mathbf{x}} \bar{\Phi}'(\mathbf{x}) \left[-\hat{G}_\Lambda \underbrace{-iT(\mathbf{x})\hat{\epsilon}T(\mathbf{x})^{-1} + \frac{\mathbb{A}_\mu^2}{2m}}_{\hat{O}_1} + \underbrace{\frac{\{\hat{p}_\mu, \mathbb{A}_\mu\}}{2m}}_{\hat{O}_2}\right]\Phi'(\mathbf{x}). \end{aligned} \tag{21}$$

Here, we introduced $\mathbb{A}_\mu = T(\mathbf{x})(-i\nabla_\mu + \mathbf{A}_{\tau_z,\mu})T^{-1}(\mathbf{x})$, $G_\Lambda^{-1}(\epsilon_n, \mathbf{p}) = i\epsilon_n - \mathbf{p}^2/2m + \mu + i\mathrm{sign}(\epsilon_n)/2\tau$, and $\mu, \nu = 1, \ldots, d$

Gradient expansion then leads to

$$\begin{aligned} S &= -\frac{1}{2}\mathrm{Tr}\,\ln(\hat{G}_\Lambda^{-1} - \hat{O}_1 - \hat{O}_2) \\ &\simeq \frac{1}{2}\mathrm{Tr}\,\ln(\hat{G}_\Lambda \hat{O}_1) + \frac{1}{4}\mathrm{tr}\,(\hat{G}_\Lambda \hat{O}_2 \hat{G}_\Lambda \hat{O}_2) + \mathcal{O}(\hat{\epsilon}^2, \nabla^3). \end{aligned} \tag{22}$$

We begin by evaluating the $\mathcal{O}_2$ term (i, j are multiindices for replica, Nambu, spin space here)

$$\begin{aligned} S_{\mathcal{O}_2} &\simeq \frac{1}{4}\int_{\mathbf{p},\mathbf{x}} v_\mu v_\nu G_\Lambda(\epsilon_n, \mathbf{p}) G_\Lambda(\epsilon_{n+m}, \mathbf{p}) \mathbb{A}^{ij,\nu}_{n,n+m}(\mathbf{x}) \mathbb{A}^{ji,\mu}_{n+m,n}(\mathbf{x}) \left[\underbrace{\theta(\epsilon_n \epsilon_{n+m})}_{RR+AA} + \underbrace{\theta(-\epsilon_n \epsilon_{n+m})}_{RA}\right] \\ &\overset{1/\tau \gg \omega_m}{\simeq} \frac{1}{4}\int_{\mathbf{p},\mathbf{x}} \frac{\mathbf{v}^2}{d} G_\Lambda(\epsilon_n, \mathbf{p}) G_\Lambda(\epsilon_{n+m}, \mathbf{p}) \mathbb{A}^{ij,\mu}_{n,n+m}(\mathbf{x}) \mathbb{A}^{ji,\mu}_{n+m,n}(\mathbf{x}) \theta(-\epsilon_n \epsilon_{n+m}) \\ &- \frac{1}{4m} \underbrace{\int_{\mathbf{p}} G_\Lambda(\epsilon_n, \mathbf{p})}_{-i\pi\nu \mathrm{sign}(\epsilon_n)} \int_{\mathbf{x}} \mathbb{A}^{ij,\mu}_{n,n+m}(\mathbf{x}) \mathbb{A}^{ji,\mu}_{n+m,n}(\mathbf{x}) \theta(\epsilon_n \epsilon_{n+m}) \end{aligned} \tag{23}$$

In the last line, we used that for the RR + AA diagrams, we could approximate

$$v_\nu G_\Lambda(\epsilon_n, \mathbf{p}) G_\Lambda(\epsilon_{n+m}, \mathbf{p}) \simeq v_\nu G_\Lambda(\epsilon_n, \mathbf{p})^2 = \partial_{p_\nu} G_\Lambda(\epsilon_n, \mathbf{p}),$$

and a partial integration.

On the other hand, we may directly employ the MF SCBA equation, Eq. (14a), to get

$$S_{\mathcal{O}_1} = \frac{1}{2}\int_{\mathbf{x}} [\underbrace{\hat{G}_\Lambda(\epsilon_n)]_{\mathbf{x},\mathbf{x}}}_{=-i\pi\nu \mathrm{sign}(\epsilon_n)} [-iT\hat{\epsilon}T^T + \frac{\vec{\mathbb{A}}^2}{2m}]^{ii}_{n,n} \tag{24}$$

We thus find

$$\begin{aligned} S_{\mathcal{O}_1} + S_{\mathcal{O}_2} &= -\frac{\pi\nu}{2}\int_{\mathbf{x}} \mathrm{sign}(\epsilon_n) \left([T\hat{\epsilon}T^T]^{ii}_{n,n} + \underbrace{i\frac{\vec{\mathbb{A}}^{ij}_{n,n+m}(1 - \theta(\epsilon_n \epsilon_{n+m}))\vec{\mathbb{A}}^{ji}_{n+m,n}}{2m}}_{\Rightarrow 0} \right) \\ &+ \int_{\mathbf{x}} \frac{1}{4d} \underbrace{\int_{\mathbf{p}} \frac{v^2}{(\mathbf{p}^2/2m - \mu)^2 + (1/2\tau)^2}}_{2\pi\nu v_F^2 \tau} \int_{\mathbf{x}} \vec{\mathbb{A}}^{ij}_{n,n'} \vec{\mathbb{A}}^{ji}_{n',n} \theta(-\epsilon_n \epsilon_{n'}) \end{aligned} \tag{25}$$

The vanishing second term in the round brackets follows from the properties of the trace

$$\begin{aligned} \mathrm{sign}(\epsilon_n)\vec{\mathbb{A}}^{ij}_{n,n'}\vec{\mathbb{A}}^{ji}_{n',n}\theta(-\epsilon_n\epsilon_{n'}) &= \sum_{\pm} \mathrm{tr}\,[\Lambda\vec{\mathbb{A}}\frac{1\pm\Lambda}{2}\vec{\mathbb{A}}\frac{1\mp\Lambda}{2}] \\ &= \sum_{\pm} \mp\mathrm{tr}\,[\vec{\mathbb{A}}\frac{1\pm\Lambda}{2}\vec{\mathbb{A}}\frac{1\mp\Lambda}{2}] \\ &= 0. \end{aligned} \tag{26}$$

We finally use in the last line

$$\begin{aligned} \vec{\mathbb{A}}^{ij}_{n,n'}\vec{\mathbb{A}}^{ji}_{n',n}\theta(-\epsilon_n\epsilon_{n'}) &= \sum_{\pm} \mathrm{tr}\,[\vec{\mathbb{A}}\frac{1\pm\Lambda}{2}\vec{\mathbb{A}}\frac{1\mp\Lambda}{2}] \\ &= \frac{1}{2}\mathrm{tr}\,[\vec{\mathbb{A}}^2 - \vec{\mathbb{A}}\Lambda\vec{\mathbb{A}}\Lambda] \\ &= -\frac{1}{4}\mathrm{tr}\,[[\vec{\mathbb{A}},\Lambda]^2] \\ &= \frac{1}{4}\mathrm{tr}\,[(D_\mu Q)^2], \end{aligned} \tag{27}$$

where

$$D_\mu Q = \partial_\mu Q + i[\hat{A}_{\tau_z,\mu}, Q]. \tag{28}$$

Thus, in total we found

$$\boxed{S = \int_{\mathbf{x}} \mathrm{tr}\,[\frac{\sigma}{32}(D_\mu Q)^2 - 2z\hat{\epsilon}Q].}_{\text{Action of Diffusive NL}\sigma\text{M}} \tag{29}$$

where $\sigma = 4\pi\nu v_F^2\tau/d$, $z = \pi\nu/4$ at the level of this calculation.

Comments

- This is the effective quantum diffusion theory valid at length scales larger than mean free path $\ell = v_F\tau$.
- From the Matsubara NLSM Eq. (29), one may obtain the Retarded/Advanced sigma model as follows:
  - keep only one retarded and one advanced Matsubara frequency called $\epsilon_{n_1}\epsilon_{n_2}$ (The matrix size of Q is then just $2_\sigma \times 2_\tau \times 2_{\mathrm{Mats.}} R$)
  - analytically continue

    $$i\hat{\epsilon} \to \begin{pmatrix} \epsilon_{n_1} & 0 \\ 0 & \epsilon_{n_2} \end{pmatrix} \to \begin{pmatrix} E_1 + i\eta & 0 \\ 0 & E_2 - i\eta \end{pmatrix} \equiv E + [\omega/2 + i\eta]\Lambda \tag{30}$$

    where $E_{1,2} = E \pm \omega/2$.

– The sigma model becomes (use $\mathrm{tr}\, Q = 0$)

$$\boxed{S = \int_{\mathbf{x}} \frac{\sigma}{32} \mathrm{tr}\,[(D_\mu Q)^2] + iz\omega \mathrm{tr}\,[\Lambda Q].} \tag{31}$$

- From $\sigma(n) = -\frac{1}{\mathcal{Z}} \frac{1}{\omega_m} \frac{\delta^2 \mathcal{Z}}{\partial A_m \partial A_{-m}}$ and setting $Q = \Lambda$, we find that the parameter σ in Eq. (29) is the physical conductivity at the mean-field (i.e. Drude) level in units of e^2/h. To be specific let's do the calculation for the RA formulation using

$$\hat{\vec{A}} = \begin{pmatrix} \vec{A}^{++} & \vec{A}^{+-} \\ \vec{A}^{-+} & \vec{A}^{--} \end{pmatrix}_{R/A}$$

$$\Rightarrow \quad \left[\hat{\vec{A}}_{\tau_z}, \Lambda\right] = 2 \begin{pmatrix} 0 & \begin{pmatrix} -\vec{A}^{+-} & 0 \\ 0 & \vec{A}^{-+} \end{pmatrix}_\tau \\ \begin{pmatrix} \vec{A}^{-+} & 0 \\ 0 & -\vec{A}^{+-} \end{pmatrix}_\tau & 0 \end{pmatrix}_{R/A} \tag{32}$$

So that

$$S[Q = \Lambda] = \sigma \int_{\mathbf{x}} \sum_\alpha \vec{A}^{+-}_\alpha \cdot \vec{A}^{-+}_\alpha. \tag{33}$$

Note that A^{++}, A^{--} don't enter, for the response we only need derivatives w.r.t. $A^{\pm,\mp}$.

- Note that this is a theory with dynamical exponent $z_Q = 2$, i.e. two spatial gradients and one frequency (= we'll see it's a diffusion theory with $D = \sigma/4z$ the diffusion constant)

- Our result for the conductivity (written in Einstein form)

$$\sigma = 2_\sigma \times 2\pi\nu D \frac{e^2}{h} \tag{34}$$

with $D = v_F^2 \tau/d$ is actually a factor of 2 too large (i.e. the parameter τ should be replaced by $\tau_{\mathrm{tr}} = \tau/2$). This mismatch with Drude theory can be cured if we also integrate out longitudinal (=Higgs) modes (which we just dropped altogether).

2.3 Soft modes (Goldstone bosons) - Wigner Dyson classes

In this section, we categorize soft modes of the sigma model manifold. We keep the model without Matsubaras to keep the notation lighter in indices.

We parametrize

$$\begin{aligned} T(\mathbf{x}) &= e^{W/2} \in G = \mathrm{Sp}(8R) \\ \Leftrightarrow W^T &= -\tau_x \sigma_y W \tau_x \sigma_y. \end{aligned} \tag{35}$$

However, only those $T \in G/H$ are relevant, i.e. we need to require

$$W\Lambda = -\Lambda W \Leftrightarrow W = \begin{pmatrix} 0 & w \\ -\bar{w} & 0 \end{pmatrix} \text{with } \bar{w}^T = \tau_x \sigma_y w \tau_x \sigma_y. \tag{36}$$

The Q field becomes

$$Q \simeq \Lambda + \Lambda W + \Lambda W^2/2 \tag{37}$$

leading to

$$S = \frac{\sigma}{32} \int (dq) \mathcal{D}^{-1}(\mathbf{q}, \omega) \mathrm{tr}\, [\Lambda W(-\mathbf{q}) \Lambda W(\mathbf{q})] \tag{38}$$

$$= \frac{\sigma}{16} \int (dq) \mathcal{D}^{-1}(\mathbf{q}, \omega) \mathrm{tr}^{\,\tau,\sigma,R} [\bar{w}(-\mathbf{q}) w(\mathbf{q})] \tag{39}$$

where

$$\left[\mathcal{D}^{-1}(\mathbf{q}, \omega)\right]^{\alpha\beta} = \mathbf{q}^2 - \frac{i\omega}{D} \tag{40}$$

A convenient parametrization is ($\sigma_a = (\mathbf{1}_\sigma, \vec{\sigma})_a$)

$$w^{\alpha\beta}(\mathbf{q}) = \frac{1}{2} \begin{pmatrix} \lambda_a \bar{d}_a^{\alpha\beta}(-\mathbf{q}) & c_a^{\alpha\beta}(\mathbf{q}) \\ \lambda_a \bar{c}_a^{\alpha\beta}(-\mathbf{q}) & d_a^{\alpha\beta}(\mathbf{q}) \end{pmatrix}_\tau \sigma_a, \tag{41a}$$

(41b)

where $\lambda_a = 1$ for $a = 0$, $\lambda_a = -1$, else. Here, the fields d are complex and $\bar{d}$ is the complex conjugate of d. Note that we have $4_{spin} \times 2_{\text{d and c}} \times R^2$ complex fields, i.e. 16 real modes, consistent with the parametrization chosen using the $16R^2$ generators of the manifold.

The parametrization is chosen such that the diagonal action is

$$\begin{aligned} S_0[d, c] &= \frac{\sigma}{16} \int_{\mathbf{q}} \sum_{\alpha\beta} \sum_{a=0}^{3} \left[\mathcal{D}^{-1}(\mathbf{q})\right]_{n_1,n_2}^{\alpha\beta} \\ &\quad \left[\bar{d}_a^{\alpha\beta} d_a^{\alpha\beta} + \bar{c}_a^{\alpha\beta} c_a^{\alpha\beta}\right]. \end{aligned} \tag{42}$$

Remarks

- The Sigma model theory is universal : all UV physics at length scales below $\ell = v_F \tau$ is washed out.

- We have a total of 8 soft modes:
  - Diffusons with Nambu space structure τ_0, τ_1
  - There are 4 Diffusons: 1 singlet ($a = 0$) and 3 triplets ($a = 1, 2, 3$)
  - Cooperons, with Nambu space structure τ_x, τ_y
  - There are again 4 of them: 1 singlet and 3 triplets.
- Each of these soft modes carries two replica indices, $\alpha\beta$.
- These Goldstone modes can also be understood diagrammatically as ladder resummations. And we'll explain it in the the next section in more details.
- breaking some of the symmetries kills some of the soft-modes:
  - Triplet soft modes are massive in the absence of spin-rotation symmetry. To see this, consider 2D and $H_{\text{Rashba}} = \alpha\hat{e}_z(\mathbf{p} \times \vec{\sigma}) = \sigma_y H^T_{\text{Rashba}}\sigma_y$, so that

$$\delta S = \int_{\mathbf{x}} \bar{\Phi} \begin{pmatrix} \alpha\hat{e}_z(\mathbf{p} \times \vec{\sigma}) & 0 \\ 0 & \hat{e}_z(\mathbf{p} \times \vec{\sigma}) \end{pmatrix} \Phi. \tag{43}$$

    Then, the symmetry group is $G = \{O|O^T\tau_x\sigma_{0,x,y,z}O\}$, i.e. $G = O(2_{\text{tau}} \times 2M \times R)$ and $H = O(2M \times R) \times O(2M \times R)$ - the NLσM fields Q don't carry any spin index any longer.
  - All cooperons are massive if TR is broken, $H \neq \sigma_y H^T \sigma_y$ and we shouldn't have gone to Nambu space. Then, the symmetry group is $G = U(2_\sigma \times 2M \times R)$ and $H = U(2_\sigma \times M \times R) \times U(2_\sigma \times M \times R)$

2.4 10-fold way: Altland-Zirnbauer classification.

We have seen above that that yes/no TR and yes/no SU(2) spin symmetry leads to three different types of NLσM - otherwise the physics beyond ℓ is universal.Here, we formalize the symmetries of the Hamiltonian and extend the universality concept from three Wigner Dyson classes to all 10 Altland-Zirnbauer classes.

The Wigner-Dyson symmetry classes are the most conventional symmetry classes and appeared first in the context of Random Matrix Theory and complex nuclei in the 1950's. The symmetry distinguishing the three Wigner-Dyson classes is time reversal symmetry. In a quantum mechanical system with spin it is realized by an antiunitary operator $T = i\sigma_y K$, thus $T^2 = -1$. However, if spin-rotation symmetry is present one may combine T with a rotation about y by $-\pi$ so that $e^{-i\pi\sigma_y/2}T = K$. Thus, in general, $T = UK$ (U unitary, K denotes complex conjugation) might square to $+\mathbf{1}$ or $-\mathbf{1}$ depending on the spin of the particles (integer or half-integer).

The Altland Zirnbauer classification is a classification of random matrices and disordered systems and thus explicitly is not categorizing systems by means of their unitary symmetries Therefore, actually, in the presence of SU(2) symmetry, one should consider Hamiltonians for spin up and spin down separately. It turns out to be equivelent to the classification of symmetric and homogeneous spaces due to Elie Cartan.

Mathematical digression: A symmetric space is a connected Riemannian manifold (M, $\langle .,. \rangle$) in which for all points $p \in M$ there is a an isometry $s_p : M \to M$, such that 1. $s(p) = p$ and 2. the corresponding differential mapping $ds_p : T_pM \to T_pM$ is the 'reflection' $ds_p = -id|_{T_pM}$. This geometric definition is isomorphic to the spaces G/H, in which there is an operation in the Lie Algebra of G which leaves the generators of the subgroup invariant and reverses all others.

This is the generic recipe:

1. Block diagonalize the Hamiltonian w.r.t. all unitary symmetries. The following recipe should then be applied to each block separately.

2. Subsequently, characterize the system by its following symmetries.

   (a) **Time reversal symmetry** $\mathcal{T} = UK$ (U unitery)

   $$\mathcal{T} : H \to UH^TU^{-1}. \tag{44}$$

   There is a total of three options: The time-reversal symmetry is absent, present and $T^2 = \mathbf{1}$, or present and $T^2 = -\mathbf{1}$.

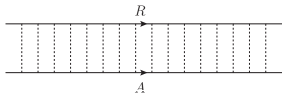
   As long as one requires the Hamiltonian to be invariant under energy shifts (i.e. somewhere deep in a band), TR is the only relevant symmetry.
   We now explain the different symmetry classes and the emergence of different soft modes in the different Wigner-Dyson classes.
   Broken time reversal symmetry: unitary class (Cartan-symbol: A). In this case, the only restriction on the Hamiltonian H is hermiticity. Therefore, the quantum mechanical evolution operator is an element of the unitary group, to which Cartan assigned the symbol A. The only soft-mode present in the diffusive regime is the retarded-advanced Diffuson

   $= \mathcal{D}^{RA}$.

   Preserved time reversal and spin rotation symmetry: orthogonal class (AI). For systems of spinless particles, the time reversal operator T squares to $+\mathbf{1}$. The most standard representation is $T = K$. Then, a time reversal invariant

H is both hermitian and symmetric, and therefore iH generates the coset space of Lie Groups $\frac{U(n)}{O(n)}$ which is denoted AI. Next to the RA-Diffuson presented above, there are also Cooperon modes present. These can be obtained from the Diffuson by applying time reversal symmetry two one of the the fermionic Green's functions, e.g. the lower (here: advanced) one:



$= \mathcal{C}^{RA}$.

Electronic systems, even though inherently spinful, can also fall into class AI: If spin is conserved, the Hamiltonian is trivial in spin-space, and reduces to the one of spinless particles. Then $\mathcal{T}$-invariance implies $H = H^T$. If, however, spin rotation invariance is broken, the system belongs to class AII, see below.
As explained above, with SU(2) spin symmetry the number of modes is 4 times larger.

Preserved time reversal and broken spin rotation symmetry: symplectic class (AII). As taught in introductory quantum mechanics courses, for spinful particles the time reversal operator squares to $-\mathbf{1}$. It can be realized by $T = i\sigma_y K$. In consequence, H is hermitain and fulfills $H = \sigma_y H^T \sigma_y$ and the coset space $\frac{U(2n)}{Sp(2n)}$ is spanned. This motivates the nomenclature "symplectic class" and is denoted by AII.

(b) **Particle Hole symmetry.** Another symmetry which is not unitary and intuitive in Quantum mechanics is charge conjugation (i.e. particle-hole symmetry) $\mathcal{Q} = VK$, (V unitary), which maps

$$\mathcal{Q} : H \to -VH^T V^{-1}. \tag{45}$$

Again, there are three possibilities: $\mathcal{Q}$ is absent, present and squares to one and present and squares to minus one, again represented by $\mathbf{1}$ or σ_y respectively.

Comments:

- Note that in the presence of $\mathcal{Q}$, the spectrum is symmetric about zero energy.
- As a simple representative to see what the manifold covered by the time evolution operator e^{iHt} is, let's consider one representative, in which $\mathcal{Q} = K$ squares to one without any other symmetries. Then $H = -H^T$, which generates the special orthogonal group (class D - Cartan actually made a distinction between)
- Particle Hole symmetry generates RR Diffusons and Cooperons, which are absent in the Wigner Dyson classes, since $G^R(\epsilon) = -V[G^A(-\epsilon)]^T V^{-1}$.

(c) **Chiral symmetry.** One has to consider the combined operator $\mathcal{C} = \mathcal{T} \circ \mathcal{Q}$, too. It maps ($W$ unitary)

$$\mathcal{C} : H \to -WHW^{-1}. \tag{46}$$

Since W is the product of U and V we see that it is either 1 or σ_y in some basis and $\mathcal{C}^2 = 1$, always.

Comment:

- A typical choice is $W = \gamma_5 = \text{diag}(\mathbf{1}, -\mathbf{1})$, in which case H is perfectly off-diagonal.
- Just as Particle Hole symmetry, chiral symmetry generates RR Diffusons and Cooperons, which are absent in the Wigner Dyson classes, since $G^R(\epsilon) = -\gamma_5 G^A(-\epsilon)\gamma_5$.

3. **The total number of symmetry classes** is thus $3_{|\mathcal{T}=0,\pm1} \times 3_{|\mathcal{Q}=0,\pm1} + 1 = 10$, where the additional 1 steps from the fact that $\mathcal{C}$ can exist or be absent with neither $\mathcal{T}$ nor $\mathcal{Q}$ present. Cartan's classification actually contains 11 classes - orthogonal matrices with even and odd dimension were considered separately (called classes D and B respectively). Of course, we also don't discuss exceptional Lie groups such as E_8.

2.5 Table of Altland-Zirnbauer symmetry classes

		time rev. T^2	chiral C	part.-hole $(CT)^2$	soft modes	compact symmetric space	compact NLσM manifold	π_1	π_2	π_3	$\delta\sigma$ 1 loop (2 loops)
Wig-	A	0	0	0	$\mathcal{D}^{RA}$	$U(N)$	$\frac{U(2N)}{U(N)\times U(N)}$	0	$\mathbb{Z}$	0	0 (WL)
ner –	AI	1	0	0	$\mathcal{D}^{RA}$, $\mathcal{C}^{RA}$	$\frac{U(N)}{O(N)}$	$\frac{Sp(4N)}{Sp(2N)\times Sp(2N)}$	0	0	0	WL
Dyson	AII	-1	0	0	$\mathcal{D}^{RA}$, $\mathcal{C}^{RA}$	$\frac{U(2N)}{Sp(2N)}$	$\frac{O(2N)}{O(N)\times O(N)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	WAL
Chiral	AIII	0	1	0	$\mathcal{D}^{RA}$,$\mathcal{D}^{RR}$	$\frac{U(p+q)}{U(p)\times U(q)}$	$U(N)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\equiv 0$
	BDI	1	1	1	$\mathcal{D}^{RA}$,$\mathcal{D}^{RR}$, $\mathcal{C}^{RA}$, $\mathcal{C}^{RR}$	$\frac{SO(p+q)}{SO(p)\times SO(q)}$	$\frac{U(2N)}{Sp(2N)}$	$\mathbb{Z}$	0	0	$\equiv 0$
	CII	-1	1	-1	$\mathcal{D}^{RA}$, $\mathcal{D}^{RR}$, $\mathcal{C}^{RA}$, $\mathcal{C}^{RR}$	$\frac{Sp(2p+2q)}{Sp(2p)\times Sp(2q)}$	$\frac{U(N)}{O(N)}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\equiv 0$
Bogo-	D	0	0	1	$\mathcal{D}^{RA}$,$\mathcal{C}^{RR}$	$SO(N)$	$\frac{O(2N)}{U(N)}$	0	$\mathbb{Z}$	0	WAL
liubov –	C	0	0	-1	$\mathcal{D}^{RA}$,$\mathcal{C}^{RR}$	$Sp(2N)$	$\frac{Sp(2N)}{U(N)}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	WL
deGen-	DIII	-1	1	1	$\mathcal{D}^{RA}$,$\mathcal{D}^{RR}$,$\mathcal{C}^{RA}$,$\mathcal{C}^{RR}$	$\frac{SO(2N)}{U(N)}$	$O(N)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	WAL
nes	CI	1	1	-1	$\mathcal{D}^{RA}$,$\mathcal{D}^{RR}$,$\mathcal{C}^{RA}$,$\mathcal{C}^{RR}$	$\frac{Sp(2N)}{U(N)}$	$Sp(2N)$	0	0	$\mathbb{Z}$	WL

1st column: Cartan symbols denoting the ten symmetry classes of Hamiltonians. The corresponding symmetric spaces are given in the 6th column. 2nd, 3rd and 4th column: absence (0) or presence (± 1) of time-reversal, chiral and particle hole symmetry respectively. As explained in 2.4 the operator realizing time-reversal, T, and charge conjugation, CT, might square to either $+1$ or -1. 5th column: possible massless soft modes. $\mathcal{D}^{RA}$ stands for a usual diffusion with one retarded and one advanced leg. Similarly $\mathcal{C}^{RA}$ for Cooperon obtained by applying TR on one of the two legs. Chiral (particle-hole) symmetry applied to one of the two legs creates a retarded-retarded diffuson $\mathcal{D}^{RR}$ (cooperon $\mathcal{C}^{RR}$). 7th column: the symmetric spaces onto which the effective field theory (NLσM) maps. Their first, second and third homotopy group are given in the following three columns. Last column: perturbative one-loop renormalization group correction to conductivity. If the one loop correction vanishes, the two loop correction is given in brackets. WL denotes weak localization, WAL weak anti-localization.

2.6 Topological terms

The table in Sec. 2.5 contains columns in which the homotopy groups of the various sigma model manifolds are listed. A non-trivial homotopy group *can* lead to the appearance of topological terms in the sigma model. I won't derive them here - they are generally related to anomalies in the fermionic quantum field theory. One example of their derivation will be given in Sec. **??**

While here we just summarize the kind of terms that can appear, we will explicitly demonstrate how they work in some exemplary cases below.

- **$\mathbb{Z}$ theta terms in a D-dimensional system.**
  - rely on $\pi_D = \mathbb{Z}$.
  - Represent the possibility of a D dimensional $\mathbf{Z}$ topological insulator
  - The action takes the form
    $$S_\theta = i\theta N[Q], \tag{47}$$
    where
    * $N[Q] \in \mathbb{Z}$ is quantized and counts the (winding-)number of instantons in Q and thereby affects the way different topological sectors are weighted in the partition sum.
    * While θ in general is not quantized.
  - If the "angle" $\theta = \pi(2n+1), n \in \mathbb{Z}$, the model is critical and localization avoided. [In only very few cases this statement can be rigorously made, otherwise it's a Haldane-like conjecture.]
  - The most prominent example is the Pruisken term
    $$S_\theta = \frac{\sigma_{xy}}{8} \int d^2 r \epsilon_{\mu\nu} \text{tr } [Q D_\mu Q D_\nu Q] = 2\pi\sigma_{xy} N[Q] \tag{48}$$
    in symmetry class A in $D = 2$.
    It's clear from differentiating w.r.t A_μ that σ_{xy} is the Hall response in units of e^2/h and $\sigma_{xy} = 1/2$ corresponds to $\theta = \pi$.
- **Wess-Zumino-Novikov-Witten (WZW) terms**
  - rely on $\pi_{D+1} = \mathbb{Z}$.
  - Occur on the surface of $D + 1$ dimensional $\mathbb{Z}$ topological insulators and protect from Anderson localization.

- The Action takes the form

$$S_{\mathrm{WZW}} = i2\pi k\Gamma[Q],$$

where

  * and $\Gamma[Q]$ measures the solid angle enclosed in a (hyper-)loop.
  * As a consequence that closing the solid angle into nord-pole or south pole is arbitrary and $\Gamma_{\mathrm{North}}[Q] - \Gamma_{\mathrm{South}}[Q] \in \mathbb{Z}$, it follows that also $k \in \mathbb{Z}$ has to be quantized (otherwise the partition function would be ill-defined).

- As a simple example, let's consider the QH system in the QH plateau $\sigma_{xx} = 0$ and $\sigma_{xy} \in \mathbb{Z}$ so that in the bulk, Eq. (48) becomes ($Q = T^{-1}\Lambda T$)

$$S = \int d^2x \frac{\sigma_{xy}}{2} \epsilon_{\mu\nu} \mathrm{tr}\,[\Lambda \partial_\mu T \partial_\nu T^{-1}], \tag{49}$$

which is a total derivative and on the boundary is

$$S = \int dy \frac{\sigma_{xy}}{2} \mathrm{tr}\,[\Lambda T \partial_y T^{-1}]. \tag{50}$$

Note that under a gauge transformation $T \to \mathrm{diag}(h_+, h_-)T$ we obtain $\delta S = \int dy \frac{\sigma_{xy}}{2} \sum_\pm \pm \mathrm{tr}\,[h_\pm \partial_y h_\pm^{-1}]$ which is a total derivative (obvious if we parametrize $h_\pm = e^{iW_\pm}$

- In $D = 2$ WZW terms can appear in the principal chiral models. For example, in class DIII the WZW-action for the orthogonal matrix fields is

$$S = \int d^2r \frac{\sigma_{xx}}{16} \mathrm{tr}\, \nabla O^T \nabla O + \frac{ik}{24\pi} \int d^2r dw \epsilon_{\mu\nu\rho} \mathrm{tr}\, \left(\tilde{O}^T \nabla_\mu \tilde{O}\right) \left(\tilde{O}^T \nabla_\nu \tilde{O}\right) \left(\tilde{O}^T \nabla_\rho \tilde{O}\right). \tag{51}$$

k is an integer called Wess-Zumino level, it has topological origin[1] and is thus unchanged under renormalization. Instead of using a gauge dependent expression as in Eq. (51), the definition of the WZW term involves introducing the extended field $\tilde{O}(\vec{r}, w)$ with the conditions $\tilde{O}(\vec{r}, w = 1) = O(\vec{r})$ and $\tilde{O}(\vec{r}, w = 0) = const.$. The value of the Wess-Zumino term (which is the integral over a total derivative) is determined by the physical plane $w = 1$ only.

- **$\mathbb{Z}_2$ theta terms**

  - rely on $\pi_D = \mathbb{Z}_2$.
  - Occur on the surface of $D+1$ dimensional $\mathbb{Z}_2$ topological insulators and protect surfaces states from Anderson localization

[1] the instanton number in the dimension above

– Similarly to the $\mathbb{Z}$ theta terms,

$$S_\theta = i\theta N[Q], \tag{52}$$

where

* $N[Q] \in \mathbb{Z}_2$ is quantized and counts the (winding-)number of instantons in Q and thereby affects the way different topological sectors are weighted in the partition sum.
* however, more like in the WZW case, for the $\mathbb{Z}_2$ case $\theta = 0, \pi$ is also quantized.

– It can be understood as descendant of the WZW term. For example, in class AII the sigma model manifold is a submanifold of the orthogonal group consisting of traceless symmetric orthogonal matrices. These additional constraints on the physical plane reduce the WZW-term to the $\mathbb{Z}_2$ theta term

$$i\pi N\left[Q\right] = \frac{ik}{24\pi}\int d^2r dw \epsilon_{\mu\nu\rho} \mathrm{tr}\left(\tilde{O}^T\nabla_\mu\tilde{O}\right)\left(\tilde{O}^T\nabla_\nu\tilde{O}\right)\left(\tilde{O}^T\nabla_\rho\tilde{O}\right)\Big|_{\tilde{O}(\vec{r},w=1)=\tilde{O}^T(\vec{r},w=1)}. \tag{53}$$

Note, that both prefactor and integral are quantized.

3 Localization physics: Quantum fluctuations beyond the saddle point

So far we have derived the effective saddle point theory. Now we go beyond by incorporating quantum corrections about the saddle point. These corrections are controlled in $1/[\sigma\ell^{D-2}] \ll 1$ and incorporate Anderson localization physics, a true quantum phenomenon.

3.1 Linear response, perturbation theory and weak localization

We return to the NLσM in retarded-advanced formulation, Eq. (31). We now want to calculate the effective quadratic action of electromagnetic source fields

$$S_{\text{eff}}[A_\mu] = -\ln[\int \mathcal{D}Q e^{-S[A,Q]}] \tag{54}$$

to leading order in fluctuations. The parameter of expansion is the dimensionless conductance, which is the stiffness of the NLSM.

For future use, we will do a slightly more general calculation: The vector potential has the property

$$\hat{A}_{\tau_z,\mu} = -\tau_x\sigma_y\hat{A}^T_{\tau_z,\mu}\tau_x\sigma_y, \tag{55}$$

which is the same property as an $Sp(2_\sigma \times 2_\tau \times 2 \times R)$ non-Abelian vector potential

$$\mathbb{A}_\mu = -\tau_x\sigma_y\mathbb{A}^T_\mu\tau_x\sigma_y. \tag{56}$$

In this case, the NLSM action is still

$$S[\mathbb{A}, Q] = \int_{\mathbf{x}} \text{tr}\,[\frac{\sigma}{32}(D_\mu Q)^2 + iz\omega\Lambda Q]. \tag{57}$$

but now $D_\mu Q = \partial_\mu Q + i[\mathbb{A}_\mu, Q]$.

We also exploit that only R/A components $A^{\pm,\mp}$ are needed for the response (see Eq. (32)) and we assume that also the generalized

$$\mathbb{A}_\mu = \left(\begin{array}{cc} 0 & \mathbb{A}^{+-}_\mu \\ \mathbb{A}^{-+}_\mu & 0 \end{array}\right) \tag{58}$$

is block-off diagonal in R/A space and moreover is a slow field as compared to Q (who's typical momenta are determined by ω).

Thus, the effective action of $\mathbb{A}$ is

$$\begin{aligned} S_{\rm eff} &= -\frac{\sigma}{32}\int_{\mathbf{x}} \langle \mathrm{tr}\,\{[\mathbb{A}_\mu, Q]^2\}\rangle \\ &+\frac{1}{2}\left(\frac{\sigma}{16}\right)^2 \int_{\mathbf{x},\mathbf{x}'} \langle \mathrm{tr}\,\{\mathbb{A}_\mu [Q,\partial_\mu Q]\}_{\mathbf{x}} \mathrm{tr}\,\{\mathbb{A}_\nu [Q,\partial_\nu Q]\}_{\mathbf{x}'}\rangle \end{aligned} \tag{59}$$

As above, Eq. (37), we expand $Q = \Lambda + \Lambda W + \Lambda W^2/2$, where $\{W,\Lambda\} = 0$. We thus immediately conclude that the second line of Eq. (59) vanishes

$$\mathrm{tr}\,\{\mathbb{A}_\mu [Q,\partial_\mu Q]\} \simeq \mathrm{tr}\,\{\underbrace{\mathbb{A}_\mu}_{\rm slow}\,\Lambda\,\underbrace{\partial_\mu W}_{\rm fast}\} - 2\mathrm{tr}\,\{\underbrace{\mathbb{A}_\mu}_{\rm off-diagonal}\;\underbrace{W\partial_\mu W}_{\rm diagonal}\} \to 0. \tag{60}$$

We thus only need to concentrate on the first line of Eq. (37), which is

$$\begin{aligned} \delta S_{\rm eff} &\equiv S_{\rm eff} - S_{\rm eff}[Q=\Lambda] \\ &= -\langle\frac{\sigma}{16}\int_{\mathbf{x}} \mathrm{tr}\,\{(\vec{\mathbb{A}}\Lambda W)^2 + \underbrace{(\vec{\mathbb{A}}\Lambda)^2 W^2}_{\to 0 \text{ as } R\to 0}\}\rangle \\ &= -\frac{\sigma}{8}\int_{\mathbf{x}} \langle \mathrm{tr}\,[(w\mathbb{A}_\mu^{-+})^2]\rangle \\ &= -\frac{\sigma}{32}\left\langle \int_{\mathbf{x}} \mathrm{tr}\,\left[\begin{pmatrix} \lambda_a \bar{d}_a & c_a \\ \lambda_a \bar{c}_a & d_a \end{pmatrix}^{\alpha\beta} \sigma_a \vec{\mathbb{A}}^{-+}_{\beta\alpha'} \begin{pmatrix} \lambda_{a'} \bar{d}_{a'} & c_{a'} \\ \lambda_{a'} \bar{c}_{a'} & d_{a'} \end{pmatrix}^{\alpha'\beta'} \sigma_{a'} \vec{\mathbb{A}}^{-+}_{\beta'\alpha}\right]\right\rangle \end{aligned} \tag{61}$$

We use the correlators following from Eq. (42), slightly generalizing the notation: $D_d^0(\mathbf{x},\mathbf{x})$ is a diffuson singlet, while $D_d^{1,2,3}(\mathbf{x},\mathbf{x})$ are diffuson triplets and analogically for cooperons denoted $D_c^a(\mathbf{x},\mathbf{x})$

$$\begin{aligned} \delta S_{\rm eff} = -\frac{1}{2}\int_{\mathbf{x}} \sum_{a=0}^{3}\sum_{\pm} \Big\{ & D_d^a(\mathbf{x},\mathbf{x})\lambda_a \mathrm{tr}\,[\frac{1\pm\tau_z}{2}\sigma_a \vec{\mathbb{A}}^{-+}_{\beta\alpha}\frac{1\mp\tau_z}{2}\sigma_a\vec{\mathbb{A}}^{-+}_{\beta\alpha}] \\ & + D_c^a(\mathbf{x},\mathbf{x})\lambda_a \mathrm{tr}\,[\frac{\tau_x\pm i\tau_y}{2}\sigma_a\vec{\mathbb{A}}^{-+}_{\beta\alpha}\frac{\tau_y\mp i\tau_x}{2}\sigma_a\vec{\mathbb{A}}^{-+}_{\beta\alpha}]\Big\}. \end{aligned} \tag{62}$$

We return to the specific case of perturbation theory, i.e.

$$\vec{\mathbb{A}}^{-+}_{\alpha\beta} = \delta_{\alpha\beta}\begin{pmatrix} \vec{A}^{-+}_\alpha & 0 \\ 0 & -\vec{A}^{+-}_\alpha \end{pmatrix}$$

and readily find that Diffusons drop out while Cooperons contribution as

$$\delta S[A] = \underbrace{2\sum_a \lambda_a D_c(\mathbf{x},\mathbf{x})}_{\delta\sigma}\int_{\mathbf{x}}\sum_\alpha \vec{A}^{+-}_\alpha \vec{A}^{-+}_\alpha \tag{63}$$

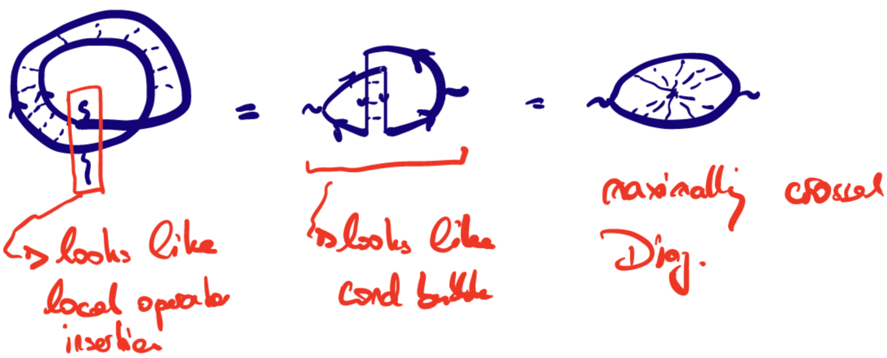


Figure 2: The calculation of the perturbative conductivity correction appears to stem from a local operator insertion $\vec{A}^2$ in the NLSM formalism, which is equivalent to maximally crossed diagrams.

In conclusion we find

$$\boxed{\sigma_{\text{Phys}}(\omega) = \sigma\left(1 + \frac{2}{\sigma}\sum_a \lambda_a D_c^a(\mathbf{x},\mathbf{x}) + \mathcal{O}(1/\sigma^2)\right)}. \tag{64}$$

Weak localization correction

We conclude with the following physical observations

- Only Cooperons contribute to $\delta\sigma$ at leading order (there is no quantum correction at leading order in perturbation theory in symmetry class A).
- The contribution of the singlet (triplet) Cooperon enhances (reduces) the conductivity.
- The integral over the Cooperons is given by ($l_\omega = \sqrt{D/\omega}$)

$$D_c^a(\mathbf{x},\mathbf{x}) = \int^{1/\ell} \frac{d^dq}{(2\pi)^d}\frac{1}{\mathbf{q}^2 - i/l_\omega^2} \simeq \begin{cases} \frac{1}{2\pi^2}[\frac{1}{\ell} - \frac{1}{l_\omega}], & d=3 \\ \frac{\ln(l_\omega/\ell)}{2\pi}, & d=2 \\ \frac{l_\omega-\ell}{2\pi}, & d=1 \end{cases} \tag{65}$$

  - In $d = 1, 2$, this integral diverges, thus the perturbative correction explodes in the dc limit $\omega \to 0$ despite the small prefactor $1/\sigma$
  - At finite temperature, we have a finite rate of phase relaxation $1/\tau_\phi \sim T^p$ (e.g. $p = 1$ for electron-electron interaction in 2D) leading to the replacement $i\omega \to 1/\tau_\phi$ (i.e. $l_\omega \to l_\phi$) at smallest frequencies.
- The parametric behavior of quantum corrections can be estimated as follows (see Fig. 3)

- The probability of a particle travelling from point A to B is $p_{A\to B} = |\sum_i A_i|^2 = \sum_i |A_i|^2 + \sum_{i\neq j} A_i^* A_j$, where A_{ij} are the possible QM amplitudes in the disordered medium.
- We represent the A_i by a line with arrow from A to B and A_i^* with reversed arrow.
- As we sum over all amplitudes, phases strongly fluctuate as a function of the disorder realization and only the semiclassical contribution appears to survive
- However, self-intersecting trajectories can have the loop going in either direction and we need to keep interference corrections as well
- Following Heisenberg, each line has a thickness λ_F, so the particle occupies $\lambda_F^{d-1} v_F dt$ at each moment in time during the loop.
- Within a certain time $t \in (\tau, \tau_\phi)$, it may access any point in a volume $(Dt)^{d/2}$, so that the integrated probability of return is

$$p_{\text{return}} = \int_\tau^{\tau_\phi} \frac{\lambda_F^{d-1} v_F dt}{(Dt)^{d/2}} \sim (E_F\tau)^{1-d} \int_1^{\tau_\phi/\tau} \frac{dt'}{(t')^{d/2}} \tag{66}$$

which has the same behavior as $D_c(\mathbf{x}, \mathbf{x})$.

- Diagramatically this corresponds to maximally crossed diagrams, see Fig. **??**.
- Particularly in the case of the 2DEG, it is interesting to study how the effect of Cooperons is killed by an external magnetic field (i.e. how the crossover from class AI or AII to A is manifested in the conductivity). The Cooperon is then the operator inverse of

$$\hat{D}_c^{a,-1} = (-i\nabla + 2e\mathbf{A})^2 + 1/l_{\phi,a}{}^2 \tag{67}$$

where the charge $2e$ reflects that Cooperons are made of two electrons and we incorporated the spin scattering rate into $1/\tau_{\phi,a} \to 1/\tau_\phi + 1/\tau_{\text{SOC}}(1-\delta_{a,0})$.

The Cooperons form 'Landau levels' with spacing $\omega_c = qB/m = 4Be \equiv 4/l_B^2$ and degeneracy $BA/(h/2e)$. Then the integral becomes a sum over Landau levels

$$\begin{aligned} D_c^a(\mathbf{x},\mathbf{x}) &= \frac{1}{A}\sum_{\mathbf{q}} \frac{1}{\mathbf{q}^2 + 1/l_{\phi,a}^2} \\ &\to \frac{B}{\pi}\sum_n \frac{1}{\omega_c(n+1/2) + 1/l_{\phi,a}^2} \\ &= \frac{1}{4\pi}\sum_n \frac{1}{n+1/2+(l_B/2l_{\phi,a})^2} \\ &= \frac{1}{4\pi}\left[\ln(n_{\max} + (l_B/2l_{\phi,a})^2) - \psi\left[(l_B/2l_{\phi,a})^2 + 1/2\right]\right] \end{aligned} \tag{68}$$

The UV cut-off is given by the condition $n_{\max}\omega_c \sim 1/\ell$ and $\psi(x)$ is the Digamma funciton.

Figure 3: Left: Trajectories joining A and B. Right: A self-intersecting loop transversed in two opposite directions.

This is typically expressed as

$$\boxed{\sigma(B) - \sigma(0) = \frac{1}{2\pi} \sum_a \lambda^a \left[\ln \left(\frac{l_B^2}{4l_{\phi,a}^2} \right) - \psi \left(\frac{l_B^2}{4l_{\phi,a}^2} + \frac{1}{2} \right) \right]}_{\text{Hikami-Larkin-Nagaoka magnetoconductance}} \tag{69}$$

Fits to this characteristic and universal magnetoconductance are a standard tool in experimental quantum transport studies to extract τ_ϕ and/or τ_{SOC}.

3.2 Strong Anderson localization and phenomenological scaling theory

The scaling theory of Anderson localization goes back to 1979 and Abrahams, Anderson, Licciardello, Ramakrishnan. They made the following hypothesis

- It's a single parameter scaling theory for the dimensionless conductance g. This means that the β function $d\ln(g)/d\ln(L)$ ought to be independent on the running IR scale (system size) L
- We know the asymptotics: $g(L) \sim \sigma L^{d-2}$ for large g and $g(L) \sim e^{-L/\xi}$ in the Anderson insulator with localization length ξ.
- The authors assumed monotonicity of the beta function (which turns out to be incorrect in some cases - see below).

Then they ended up with their famous picture for the scaling, Fig. 4. We expect

- Stability of the diffusion theory in D = 3. The transition to the insulating state is expected at $g \sim 1$
- Localization of all states for $D \leq 2$, where $D = 2$ is most interesting for the Anderson transition (critical dimension).

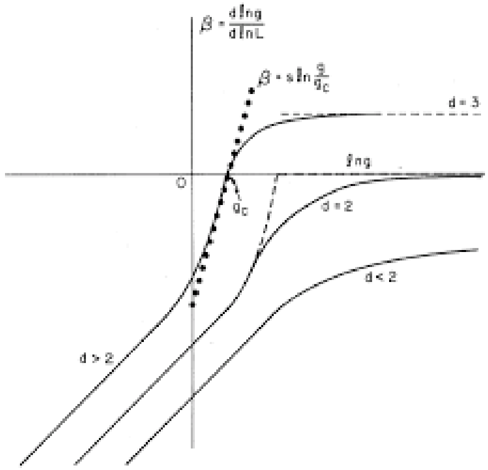


Figure 4: Phenomenological one parameter scaling theory of Anderson localization [Abrahams *et al.* PRL 42, p. 673 (1979)].

- The NLSM technique allows to calculate this beta function starting from large g at or near the critical dimension $D = 2$.
- We will also consider smaller D and will see that non-trivial phases may appear in this case as well.

3.3 2D: Renormalization group approach

3.3.1 Generalities on the Renormalization group approach

General motivation for the renormalization group approach

- From an abstract QFT view point:
  - QFTs typically contain divergencies (high energy theorists are mostly concerned about UV divergencies).
  - These divergencies need to be regularized (e.g. by a cut-off or dimensional regularization) and counter terms are introduced in the bare action to cancel divergencies in the diagrams
  - From the high-energy perspective there are three cases
    1. *Superrenormalizable theories:* Only a finite number of diagrams is "superficially" divergent (in the UV), so that only a finite number off counter terms is needed.

2. *Renormalizable theories:* An infinite number of diagrams is diagrams is superficially divergent, but at each order in a small parameter (e.g. small coupling constant or $1/N$) only a finite number of counterterms is needed to cure these divergencies.
3. *Non-renormalizable theories:* The number of superficially divergent diagrams blows up with expansion order, so one would need an infinite number of counter terms to cure these divergencies.

I will not show this here, but mathematically inclined students might want to check Zinn-Justin's book to see the proof or renoramalizability (order by order, in the above sense) of ϕ^4 theory.

- From a condensed matter view point
  - We're always interested at the physics in the IR, and want to to derive the effective Hamiltonian/action valid at smallest energies
  - To this purpose, we integrate out Quantum/thermal flucations of short distance/short time modes.
  - The RG procedure does this iteratively. One RG step is subdivided into the following three steps
    1. Split fields into slow and fast: For a ϕ^4 theory, this looks like ($b > 1$)

$$\phi(x) = \int^{\Lambda} (dp)e^{ipx}\phi(x) = \underbrace{\int^{\Lambda/b} (dp)e^{ipx}\phi(x)}_{=:\phi_s} + \underbrace{\int_{\Lambda/b}^{\Lambda} (dp)e^{ipx}\phi(x)}_{=:\phi_f}, \tag{70}$$

if the fields have a target manifold which is a group $T(\mathbf{x}) \in G$ or a coset space (e.g. the NLσM field $Q = T^{-1}\Lambda T$)

$$T(\mathbf{x}) = T_f(\mathbf{x})T_s(\mathbf{x}). \tag{71}$$

2. Obtain the effective action of slow fields by integrating out fast fields. Using the notation

$$S[\phi_s + \phi_f] = S[\phi_s] + S[\phi_f] + S_{\text{imt}}[\phi_s, \phi_f], \tag{72}$$

this amounts to

$$S_{\text{eff}}[\phi_s] = S[\phi_s] - \ln\langle e^{-S_{\text{int}}[\phi_s,\phi_f]}\rangle_{\phi_f}. \tag{73}$$

and analogically for the NLσM case (see Sec. 3.3.2)

3. Rescaling of momenta $p \to p' = bp$, so that the new theory has the same cut-off as the original theory. Fields may also be rescaled $\phi \to \phi' = b^{d_\phi}\phi$, usually in such a way that the coupling constant of the kinetic term is not renormalized (of course group and NLσM fields Q can't be rescaled, otherwise, they loose their algebraic properties, e.g. $Q^2 = 1$.).

After one RG step, all coupling constants g_a are mapped to $g_a \mapsto g'_a(\{g\}, b)$.

- This procedure is then iteratively repeated and expressed in the form of differential RG equations.
  - * A sloppy way to get them, is to require Λ/b is infinitesimally less then Λ and so that we can expand in $\ln(b) \ll 1$, i.e.

$$g'_a(\{g\}, b) = g_a + \ln(b)\beta_a(\{g\}). \tag{74}$$

    At this level, the iterative RG procedure is equivalent to the differential equation

$$\boxed{\frac{g'_a(\{g\}, b) - g_a}{\ln(b)} \to \frac{dg_a}{d\ln(b)} = \beta_a(\{g\}).} \tag{75}$$

Renoramalization group equation

  - * A more precise way is to keep $\ln(b) \gg 1$, in such a way that only the most divergent diagrams are discriminated and kept (as mentioned, from the QFT perspective RG is a way to tame divergencies). In this case, however, the form of the discrete renormalization step has the form Eq. (74) only if the theory is renormalizable and is equivalent to the resummation of logarithmic diagrams.

- Comments regarding the solutions of RG equations
  - Those g^*_a such that $\beta_a(\{g_a\}) = 0 \forall a$ are called fixed points. Here the systems are self-similar (just as fractals).
  - In most continuous phase transitions, the following linearization

$$\beta_a(\{g^* + \delta g\}) = \partial_{g_b}\beta_a(\{g^*\})\delta g_b \tag{76}$$

    leads to a non-zero matrix $\partial_{g_b}\beta_a(\{g^*\})$ who's (left) eigenvalues λ_α define the scaling dimension of certain coupling constants with associated operators $\hat{O}_\alpha$ at the phase transition. We distinguish

    - * $\lambda_\alpha > 0$: The flow is directed away from the fixed point ($\hat{O}_\alpha$ is a "relevent operator").
    - * $\lambda_\alpha = 0$: The flow is unchanged, $\hat{O}_\alpha$ is a "marginal operator" and the fixed point is actually part of a line of fixed points.
    - * $\lambda_\alpha < 0$: The flow is towards the fixed point ("irrelevant operator")

    (in the condensed matter convention, the direction of flow always is towards the infrared).
  - This allows to distinguish different fixed points
    - * All operators are irrelevant: A stable, attractive fixed point, i.e. a stable phase.

  * If one (or more) operators is relevant: repulsive fixed point, i.e. a phase transition.
- Technically, most RG calculations are performed perturbatively perturbing about the free fix point $g_a = 0 \forall a$.

3.3.2 Renormalization group flow of non-linear sigma model in the Wigner Dyson classes

We have seen, Eq. (64), that the perturbative corrections to conductivity diverge logarithmically in 2D and that RG theory is a method to tame these logarithmic divergencies. Here we apply the strategy to the NLσM of class AI.

Splitting of fast and slow fields. As mentioned above we split $T(\mathbf{x}) = T_f(\mathbf{x})T_s(\mathbf{x})$, so that

$$S[Q] = \frac{\sigma}{32}\int_{\mathbf{x}} \mathrm{tr}\,[(\partial_\mu Q_f + i[\mathbb{A}(\mathbf{x}), Q_f])^2], \tag{77}$$

with $\mathbb{A}(\mathbf{x}) = -iT_s\partial_\mu T_s^{-1}$.

RG step. We have already done the calculation of calculating the effective theory of slow fields above! We can simply return to Eq. (62) and assume $D_c^a(x,x) = D_d^a(x,x) = D(x,x)$ are now all integrals over fast fields, so that

$$\begin{aligned}
\delta S_{\text{eff}} &= -\frac{D(\mathbf{x},\mathbf{x})}{4}\sum_{a,b}\lambda^a(-1)^{b=z}\int_{\mathbf{x}}\Big\{\mathrm{tr}\,[\frac{1\pm\tau_z}{2}\sigma_a\vec{\mathbb{A}}^{-+}_{\beta\alpha}\frac{1\mp\tau_z}{2}\sigma_a\vec{\mathbb{A}}^{-+}_{\beta\alpha}] \\
&\qquad + \mathrm{tr}\,[\frac{\tau_x\pm i\tau_y}{2}\sigma_a\vec{\mathbb{A}}^{-+}_{\beta\alpha}\frac{\tau_y\mp i\tau_x}{2}\sigma_a\vec{\mathbb{A}}^{-+}_{\beta\alpha}]\Big\} \\
&= -\frac{D(\mathbf{x},\mathbf{x})}{4}\sum_{a,b}\int_{\mathbf{x}}\mathrm{tr}\,[(\tau_b\sigma_a)^T(\sigma_y\tau_x\mathbb{A}^{-+}_{\beta\alpha})(\tau_b\sigma_a)(\mathbb{A}^{-+}_{\beta\alpha}\sigma_y\tau_x)] \\
&\overset{(1)}{=} -D(\mathbf{x},\mathbf{x})\int_{\mathbf{x}}[\sigma_y\tau_x\mathbb{A}^{-+}]_{lk,\beta\alpha}[\mathbb{A}^{-+}\sigma_y\tau_x]_{lk,\beta\alpha} \\
&\overset{(2)}{=} -D(\mathbf{x},\mathbf{x})\int_{\mathbf{x}}\mathrm{tr}\,[(\sigma_y\tau_x\mathbb{A}^{-+})(\mathbb{A}^{-+}\sigma_y\tau_x)] \\
&= -\frac{4D(\mathbf{x},\mathbf{x})}{32}\int_{\mathbf{x}}\mathrm{tr}\,[(\nabla Q_s)^2]
\end{aligned} \tag{78}$$

At (1), we used the Fierz-identity of SU(4)

$$\sum_{ab}[\sigma_a\tau_b]_{ij}[\sigma_a\tau_b]_{kl} = 4\delta_{il}\delta_{jk} \tag{79}$$

and at (2)

$$[\sigma_y\tau_x\mathbb{A}^{-+}]_{lk,\beta\alpha} = -[\mathbb{A}^{+-T}\sigma_y\tau_x]_{lk,\beta\alpha} = [\sigma_y\tau_x\mathbb{A}^{-+}]_{kl,\alpha\beta} \tag{80}$$

Rescaling

We found

$$S[Q_s] = \frac{\sigma - 4D(\mathbf{x},\mathbf{x})}{32}\int d^dx \mathrm{tr}\left[(\partial_\mu Q_s)^2\right] \tag{81}$$

The rescaling of momenta $p \to p' = bp$ (or $x \to x' = x/b$) implies a final expression for the renormalized coupling constant

$$\sigma \to \sigma' = b^{d-2}[\sigma - 4D(\mathbf{x},\mathbf{x})]. \tag{82}$$

RG equation

We are most interested in the vicinity of 2 dimensions. In exactly 2D, the integral

$$D(\mathbf{x},\mathbf{x}) = \int_{\Lambda/b}^{\Lambda}\frac{d^2p}{(2\pi)^d}\frac{1}{p^2+L^{-2}} \simeq \frac{1}{2\pi}\ln(b). \tag{83}$$

A more formal calculation (dimensional regularization and d near 2) leads to essentially the same result.

Following the above mentioned strategy, this leads to the following one-loop renormalization group equation

$$\boxed{\frac{d\sigma}{d\ln(b)} = (d-2)\sigma - \frac{2}{\pi}.} \tag{84}$$

RG equation of NLσM in class AI

Comments:

- At $d = 2+\epsilon$ there are three fixed points
  1. The diffusive metal, $\sigma^* = \infty$, which is the attractive fixed point we were expanding about.
  2. In it's vicinity, a repulsive fixed point $\sigma^* = 2/(\pi\epsilon)$ of the Anderson-metal insulator transition
  3. An attractive fixed point $\sigma^* = 0$, outside the regime of control, representing the insulator.
- Note that the bare value of $\sigma \sim E_F\tau(E_F)^{(d-2)/2}$, so that there is a *mobility edge* E^* (i.e. an energy below which all states are localized)

$$E^* \sim (\sigma^*)^{\frac{2}{d}}. \tag{85}$$

- Critical exponents of the Anderson transition are defined as

$$\begin{aligned} \xi &\sim (E_* - E)^{-\nu}, \text{localization length exponent} \quad (86) \\ \sigma &\sim (E - E_*)^s, \text{conductivity exponent} \quad (87) \end{aligned}$$

  with scaling relation $s = \nu(d-2)$. Expansion near the transition yields $\beta'(\sigma^*) = \epsilon$ and thus $\delta\sigma \sim (L/\ell)^\epsilon \delta\sigma_0$. The correction is large when $L \sim \ell(\delta\sigma_0)^{-1/\epsilon}$ which defines $\xi \sim (E_* - E_F)^{-1/\epsilon}$, i.e. $\nu = 1/\epsilon$. (The numerical value in 3D is $\nu \sim 1.57$).

- The flow towards strong coupling is typical for compact order parameter manifolds in 2D and is a manifestation of Mermin-Wagner-theorem. Here, it signals strong localization, in accordance with Fig. 4.

- One may also estimate the localization length ξ_{loc}, which is the length scale at which the solution to the RG equation $\sigma(L) = \sigma_{\text{Drude}} - 2\ln(L/\ell)/\pi$ vanishes:

$$\xi_{\text{loc}} \sim \ell e^{\pi\sigma_{\text{Drude}}/2} \quad (88)$$

  Obviously, this scale is exponentially large.

- We didn't discuss the frequency term, but z is actually not renormalized in 2D.

- We briefly comment on the RG flow in 2D in the other Wigner-Dyson symmetry classes (without calculation, just reading off Eq. (64))
  - In class A, there are no quantum corrections at one loop order, $d\sigma/d\ln(L/\ell) = \mathcal{O}(1/\sigma)$
  - In class AII, the equation is $d\sigma/d\ln(L/l) = 1/\pi$.
    * This renormalization group flow appears to violate Mermin Wagner. The culpit is the Replica limit (for any $R \in \mathbb{N}$ the beta function is non-negative, in accordance with the theorem.
    * The Weak anti-localization stabilizes the diffusive metallic phase $\sigma = \infty$. Since $\sigma = 0$ (the Anderson insulator) is also a stable fix point, there must be a phase transition at some intermediate σ.

3.3.3 Scaling theory of the integer Quantum Hall transition

In this section we discuss a first example of the interplay of localization physics with topology and study the NLσM describing the states inside a (disorder broadened) Landau level and theoretically motivating the scaling flow at the QH transtion, see Fig 5.

$$S = \int d^2x \frac{\sigma_{xx}^{(0)}}{8} \text{tr}\,[(D_\mu Q)^2] + \frac{\sigma_{xy}^{(0)}}{8} \epsilon_{\mu\nu} \text{tr}\,[Q D_\mu Q D_\nu Q]. \quad (89)$$

Here the superscript $^{(0)}$ indicates the bare (Drude) value)

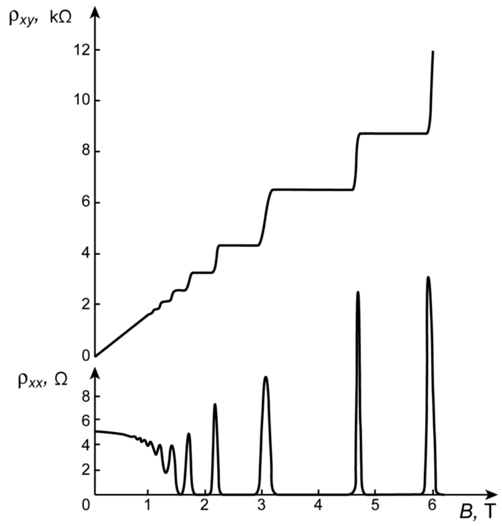


Figure 5: The Quantum Hall effect.

- In this case $Q \in \frac{U(2R)}{U(R)\times U(R)}$, note that $\pi_2(\frac{U(2R)}{U(R)\times U(R)}) = \mathbb{Z}$.
- The term proportional to σ_{xy} countes the number of instantons. A single instanton occurs in a single replica, denoted α_0, here, and is given by

$$[Q_{\text{inst}}]_{\alpha\beta} = \delta_{\alpha\beta}\delta_{\alpha,\alpha_0}\hat{n}(\mathbf{x}) \cdot (\Lambda_x, \Lambda_y, \Lambda_z) + (1 - \delta_{\alpha,\alpha_0})\delta_{\alpha\beta}\Lambda_z \tag{90}$$

and

$$\hat{n}(\mathbf{x}) = \frac{(2\lambda x, 2\lambda y, \mathbf{x}^2 - \lambda^2)^T}{\lambda^2 + \mathbf{x}^2} \tag{91}$$

is a unit vector containing one Skyrmion of size λ, see Fig. 6.

- Insertion of a single skyrmion leads to an action

$$S[Q_{\text{inst}}] = \underbrace{2\pi\sigma_{xx}^{(0)}}_{\text{price from kinetic energy}} + 2\pi i\sigma_{xy}^{(0)} \tag{92}$$

- To account for skyrmions, the path integral is evaluated as

$$\mathcal{Z} = \sum_W \mathcal{Z}_W \quad \mathcal{Z}_W = \int \mathcal{D}T \exp[-S[T^{-1}Q_{\text{inst}}^{(W)}T]] \tag{93}$$

where W is the winding number of the instanton. Note that the soft modes T which smoothly distort the instanton include so called "zero modes", i.e. transformations of Q_{inst} which do not cost any extra energy (e.g. the position of the defect in real and replica space).

- As above, $\delta^2 Z/\delta A_\mu^{+-}\delta A_\nu^{-+}$ defines the conductivity $\sigma_{\mu\nu}$. Note that σ_{xy} is odd under reflection of B-field, while σ_{xx} is even.
- The evaluation of the physical conductivity after incorporating all quantum fluctuations has formally the form

$$\sigma_{xx} = \sigma_{xx}^{(0)} + \sum_{W=0}^{\infty} \cos(2\pi W \sigma_{xy}^0) c_W(\sigma_{xx}^{(0)}) \tag{94}$$

$$\sigma_{xy} = \sigma_{xy}^{(0)} + \sum_{W=0}^{\infty} \sin(2\pi W \sigma_{xy}^0) c'_W(\sigma_{xx}^{(0)}), \tag{95}$$

  where the coefficients c_W, c'_W account for all fluctuations in a given topological sector with W (anti-)skyrmions.
- Pruisken et al. have calculated these fluctuation corrections perturbatively and recast it in RG language using minimal subtraction scheme. In addition to the usual integration of fast non-topological modes, a non-standard mode of integration is the size of the skyrmion: Instantons with skyrmion size $\lambda \in (1/\Lambda, b/\Lambda)$ are also integrated out as fast modes. This leads to

$$\frac{d\sigma_{xx}}{d\ln(b)} = -\frac{1}{2\pi^2\sigma_{xx}} \qquad -c\sigma_{xx}e^{-2\pi\sigma_{xx}}\cos(2\pi\sigma_{xy}) \tag{96a}$$

$$\frac{d\sigma_{xy}}{d\ln(b)} = \qquad c\sigma_{xx}e^{-2\pi\sigma_{xx}}\sin(2\pi\sigma_{xy}) \tag{96b}$$

  with some $c > 0$. The corresponding RG flow is plotted in Fig. 6, right. The overall topology of the flow (associated to the name of D. Khmelnitskii) has been confirmed experimentally.
- For a discussion of subtleties of this procedure, see the book by Altland and Simons, p.535.

3.3.4 Disordered Dirac fermions I: RG flow and Avoided localization

In the previous section, we saw how a topological term, the θ-term, is responsible for the transition between two topological distinct bulk insulators (in that case two different QH states).

Now we will study the surface of a topological insulator and see how Anderson localization is avoided in this case.

The model which we study falls into class AIII, where the beta function is known to be exactly zero to all orders in Perturbation theory, see Sec. 2.5, even for non-topological

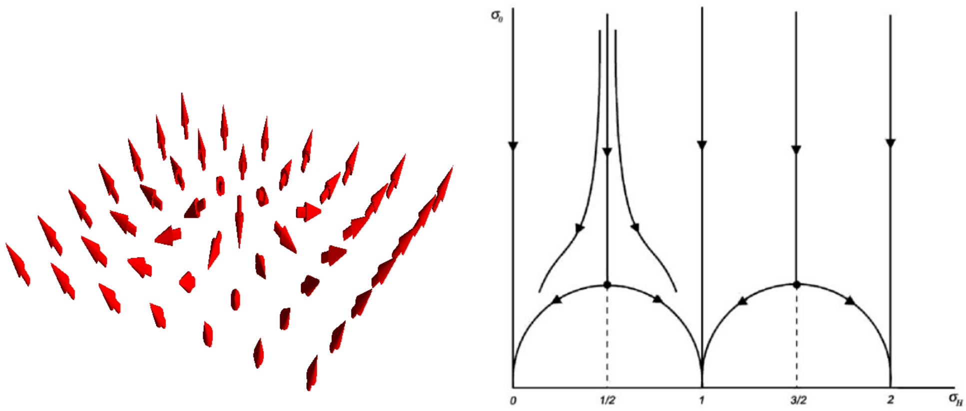


Figure 6: Left:A Skyrmion field, Eq. (91). Right: Schematic flow of Eqs. (96).

systems. At the end of this section, we therefore also comment on the impact of th WZNW term in the other two 'principle chiral models' of class DIII (NLSM manifold = orthogonal group, antilocalization) and CI (NLSM manifold = symplectic group, localizing).

Before studying the RG flow, we first present an alternative way to derive the NLSM and introduce the technique of non-Abelian Bosonization.

Non-Abelian Bosonization - Generalities We first consider a free fermion problem with N flavors

$$\mathcal{L} = \bar{\psi}[-\underbrace{E + i\eta}_{i\epsilon_n \to E+i\eta} + v \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix}]\psi. \tag{97}$$

We're interested in states at $E = 0$, and moreover set $v = 1$ for simplicity . Then, this Lagrangian has a $U(N)_L \times U(N)_R$ symmetry:

$$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \to \begin{pmatrix} U_R\psi_\uparrow \\ U_L\psi_\downarrow \end{pmatrix}, \tag{98}$$

$$\bar{\psi} = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow) \to \left(\bar{\psi}_\uparrow U_L^\dagger, \bar{\psi}_\downarrow U_R^\dagger\right) \tag{99}$$

where each of $\psi_\uparrow$ and $\psi_\downarrow$ is an N spinor on its own, and U_R, U_L are independent, unitary matrices.

To obtain the corresponding classical Noether currents, we consider space-dependent rotations

$$\mathcal{L} \to \sqrt{2}\bar{\psi} \begin{pmatrix} 0 & -i\partial_z - iU_L^\dagger \partial_z U_L \\ -i\partial_{\bar{z}} - iU_R^\dagger \partial_{\bar{z}} U_R & 0 \end{pmatrix} \psi. \tag{100}$$

We use the notation $\partial_z = \frac{1}{\sqrt{2}}[\partial_x - i\partial_y]$ and $z = \frac{x+iy}{2}$.

- Classically, there are two sets of conserved current vectors

$$j_{\mu,i} = \bar{\psi}\sigma_\mu t_i \psi \tag{101}$$
$$j^5_{\mu,i} = \bar{\psi}\sigma_z\sigma_\mu t_i \psi = i\epsilon_{\mu\nu} j_{\nu,i} \tag{102}$$

Here $t_i \in \mathfrak{u}(k)$ normalizad as $\mathrm{tr}\,[t_i, t_j] = 2\delta_{ij}$

- It will not be important for what follows, but the quantum anomaly implies equations of motion of the kind

$$\partial_\mu j_{\mu,i} = 0 \tag{103}$$
$$\partial_\mu j^5_{\mu,i} = \frac{e}{2\pi}\epsilon^{\mu\nu}\mathrm{tr}\,[F_{\mu\nu}] \tag{104}$$

in the presence of a non-trivial magnetic field.

- Instead of the axial and conserved currents consider the chiral currents

$$j = \psi_\uparrow \otimes \bar{\psi}_\downarrow \tag{105}$$
$$\bar{j} = \psi_\downarrow \otimes \bar{\psi}_\uparrow \tag{106}$$

with corresponding Noether conservation laws

$$\partial_z \bar{j} = 0 = \partial_{\bar{z}} j. \tag{107}$$

Note that I use a matrix notation and one may readily extract $j_i = \mathrm{tr}\,[t_i j]$ using U(N) generators.

- It's evident that under a $U(N) \times U(N)$ transformation

$$j = \psi_\uparrow \bar{\psi}_\downarrow \to U_R j U_R^\dagger \tag{108}$$
$$\bar{j} = \psi_\downarrow \bar{\psi}_\uparrow \to U_L \bar{j} U_L^\dagger \tag{109}$$

- Now we follow Witten in an educated guess for the bosonized theory. We require

  1. The bosonic target manifold should be invariant under $U(N) \times U(N)$.
  2. Left and right currents should transform correctly in the bosonized theory
  3. And fulfill the correct equations of motion ("current algebra")

  We now move to fulfilling these conditions

  1. Manifolds with the first property are group manifolds, in our case $U(N)$, where $U \to U_L U U_R^{-1}$.

2. The correct transformation behavior is fulfilled by the bosonization dictionary

$$j = \psi_\uparrow \otimes \bar{\psi}_\downarrow \leftrightarrow \frac{1}{\sqrt{8\pi}} U^\dagger \partial_z U \tag{110a}$$

$$\bar{j} = \psi_\downarrow \otimes \bar{\psi}_\uparrow \leftrightarrow \frac{1}{\sqrt{8\pi}} U \partial_{\bar{z}} U^\dagger, \tag{110b}$$

$$\psi_\uparrow \otimes \bar{\psi}_\uparrow \leftrightarrow -\Lambda U^\dagger \tag{110c}$$

$$\psi_\downarrow \otimes \bar{\psi}_\downarrow \leftrightarrow \Lambda U. \tag{110d}$$

By power counting, it's clear that Λ is an energy (of the order of the UV cut-off).

3. To find the action that is consistent with Noether theorem, make the Ansatz

$$S[U] = \frac{\sigma}{8\pi} \int_{\mathbf{x}} \mathrm{tr}\,[\partial_\mu U \partial_\mu U^\dagger] - \frac{ik}{12\pi} \int_{(\mathbf{x},w)\in\mathbb{R}^2\times[0,1]} \epsilon_{\mu\nu\rho} \mathrm{tr}\,[\tilde{U}^\dagger \partial_\mu \tilde{U} \tilde{U}^\dagger \partial_\nu \tilde{U} \tilde{U}^\dagger \partial_\rho \tilde{U}] \tag{111}$$

Comments

– Here $\tilde{U}(\mathbf{x}, w)$ is defined by $\tilde{U}(\mathbf{x}, 0) = U$ and $\tilde{U}(\mathbf{x}, 1) = \mathbf{1}$. The prefactor k is unknown , but $k \in \mathbb{Z}$ for topological reasons .
– A priori, $\sigma > 0$ is another unknown constant.
– We now derive the equations of motion of currents/conservation laws. We use the following intermediate result (left as an excercise)

$$S[U_l U_r] = S[U_l] + S[U_r] + \frac{1}{4\pi} \int_{\mathbf{x}} \mathrm{tr}\,[U_r \partial_\mu U_r^\dagger (\sigma \delta_{\mu\nu} - ik\epsilon_{\mu\nu}) U_l^\dagger \partial_\nu U_l]. \tag{112}$$

The conservation laws follow from slow rotation of the bosonized field U in the same way as the usual Noether theorem. After a bit of algebra (also left as an excercise)

$$U \to U_L(\mathbf{x}) U : (\sigma + k)\partial_z (U \partial_{\bar{z}} U^\dagger) + (\sigma - k)\partial_{\bar{z}} (U \partial_z U^\dagger) = 0 \tag{113}$$

$$U \to U U_R(\mathbf{x}) : (\sigma + k)\partial_{\bar{z}} (U^\dagger \partial_z U) + (\sigma - k)\partial_z (U^\dagger \partial_{\bar{z}} U) = 0 \tag{114}$$

Consistency with the fermionic conservation laws imposes $\sigma = k$.
– What's left to fix is the value of σ. The simplest short-cut to use $U = e^{i\phi t_0} V$, $V \in SU(N)$ and compare the action of ϕ from what's known from Abelian bosonization (differently stated: We simply evaluate correlators of $\mathrm{tr}\,[j] \sim \partial_z \phi$ in both fermionic and bosonic language). Then we readily see that $\sigma = k = 1$.

This concludes the "derivation" (better: motivation) of the non-Abelian bosonization rules, Eq. (110), where fermionic correlators are evaluated w.r.t Eq. (97) and bosonic correlators are evaluated w.r.t. Eq. (111) at $\sigma = k = 1$.

Derivation of NLσM via non-Abelian Bosonization We now study a localization problem of class AIII, which is describes the surface of a 3D TI with bulk winding number k (i.e. with k Dirac nodes on the surface).

$$\mathcal{L} = \bar{\psi}(\mathbf{p} - \mathbf{A}) \cdot \vec{\sigma}\psi, \tag{115}$$

where ψ is a $2k$ spinor and we expand

$$\mathbf{A} = \mathbf{A}_i t_i, \quad t_i \in \mathfrak{u}(k). \tag{116}$$

The random $U(N)$ vector potential is Gaussian white noise distributed

$$\langle A_i^\mu(\mathbf{x}) A_j^\nu(\mathbf{x}')\rangle_{\text{dis}} = \lambda_i \delta_{ij}\delta_{\mu\nu}\delta(\mathbf{x} - \mathbf{x}'), \tag{117}$$

$\lambda_0 = \lambda_A$ and $\lambda_{i\neq 0} = \lambda$ describe the $U(1)$ and $SU(N)$ disorder strength.

We use the bosonic representation of this model following the previously derived non-Abelian Bosonziation. As a first step, we extend to a replicated theory, i.e. ψ is now a $2kR$ spinor.

The gauged WZNW is thus (for non-topological gauge field configurations)

$$\begin{aligned} S[U, A] &= S[U] \\ &+ \frac{1}{4\pi}\int_{\mathbf{x}} (\delta_{\mu\nu} - i\epsilon_{\mu\nu})\{\underbrace{i\text{tr}\,[A_\mu U^\dagger \partial_\nu U] + i\text{tr}\,[U\partial_\mu U^\dagger A_\nu]}_{\to S_1} - \underbrace{\frac{1}{2}\text{tr}\,[[U, A_\mu][U^\dagger, A_\nu]]}_{\to S_2}\} \end{aligned} \tag{118}$$

Note that $U \in U(kR)$, but A_μ are $U(k)$ vector potentials ($A_\mu = A_{\mu,i} t_i \otimes \mathbf{1}_{\text{replicas}}$).

We start by studing the most relevant term S_2 with two gauge fields. While formally, one should take full field integral, it is sufficient to consider the quadratic term to leading order to see that it generates a mass for some of the NLσM modes.

$$\begin{aligned} \langle S_2 \rangle &= -\frac{1}{4\pi}\int_{\mathbf{x}} \lambda_i \delta(0) \text{tr}\,[[U, t_i][U^\dagger, t_i]] \\ &\overset{U=e^{iW}}{\simeq} \frac{\lambda}{4\pi}\int_{\mathbf{x}} \delta(0) \sum_{t_i \in \mathfrak{su}(k)} \text{tr}\,[(i[W, t_i])^2] \geq 0. \end{aligned} \tag{119}$$

Here, $\delta(0)$ is the delta function evaluated at coincident points using an appropriate UV regularization. We used the exponential map to parametrize the unitary group. Note that W, t_i are both hermitian, and thus $i[W, t_i]$ is also hermitian. By consequence $(i[W, t_i])^2$ is a positive definite matrix. By consequence $\langle S_2 \rangle$ is minimized by those W which commute with all $t_i \in \mathfrak{su}(k)$, i.e. when W is only a matrix in replicas.

We therefore have to restricting ourselves to $U \in U(R)$ and we find that the only contribution to S_1 stems from $A_{\mu,i} = 0$. We use that

$$S_1|_{A_0} = \frac{1}{2\pi} \int_{\mathbf{x}} A_\mu \epsilon_{\mu\nu} \mathrm{tr}\,[t_0(U^\dagger \partial_\mu U)] \tag{120}$$

and thus

$$\begin{aligned} \delta S_{\text{eff}} &= -\ln\langle e^{-S_1} \rangle \\ &= \frac{\lambda_A}{8\pi^2} \int_{\mathbf{x}} (\mathrm{tr}\,[t_0 U^\dagger \partial_\mu U])^2 \\ &= -\int_{\mathbf{x}} \frac{c}{8\pi} (\mathrm{tr}\,[U^\dagger \partial_\mu U])^2 \end{aligned} \tag{121}$$

Here, tr goes over the kR dimensional space and $t_0 = \sqrt{2/k}\mathbf{1}$. In Eq. (122) we switched to the notation where tr only goes over R dimensional space, hence the result for $c = (\lambda_A k^2/\pi) \times (2/k)$, where the last factor stems from the normalization of t_0 .

The effective action after averaging has thus the form

$$S[U] = \int_{\mathbf{x}} \frac{\sigma}{8\pi} \mathrm{tr}\,[\partial_\mu U \partial_\mu U^\dagger] - \frac{c}{8\pi} (\mathrm{tr}\,[U^\dagger \partial_\mu U])^2 - \frac{ik}{12\pi} \int_{(\mathbf{x},w)} \epsilon_{\mu\nu\rho} \mathrm{tr}\,[\tilde{U}^\dagger \partial_\mu \tilde{U} \tilde{U}^\dagger \partial_\nu \tilde{U} \tilde{U}^\dagger \partial_\rho \tilde{U}] \tag{122}$$

where $\sigma = k$ at the UV and $c = 2\lambda_A k/\pi$. The term propotional to c is specific to chiral classes but not to Dirac fermions.

Renormalization group of NLσM in class AIII with WZW term We now consider Eq. (122) at arbitrary values of $\sigma, c \in \mathbb{R}$ and $k \in \mathbb{Z}$ and calculate the RG equations for $\sigma \gg 1$. We first split

$$U = e^{i\phi t_0} V. \tag{123}$$

Note that $V \in SU(R), t_0 = \sqrt{2/R}\mathbf{1}_R$ (We have switched the notation from t_i being generators of SU(k) to t_i being generators of SU(R)). Then

$$\begin{aligned} S[\phi, V] &= \int_{\mathbf{x}} \frac{\sigma + Rc}{4\pi} (\partial_\mu \phi)^2 + S[V], \\ S[V] &= \int_{\mathbf{x}} \frac{\sigma}{8\pi} \mathrm{tr}\,[\partial_\mu V \partial_\mu V^\dagger] - \frac{ik}{12\pi} \int_{(\mathbf{x},w)} \epsilon_{\mu\nu\rho} \mathrm{tr}\,[\tilde{V}^\dagger \partial_\mu \tilde{V} \tilde{V}^\dagger \partial_\nu \tilde{V} \tilde{V}^\dagger \partial_\rho \tilde{V}] \end{aligned} \tag{124}$$

We readily see that $\sigma + Rc$ is not renormalized at any R (ϕ is a Gaussian field), and thus

$$\frac{d\sigma}{d\ln(b)} = -R\frac{dc}{d\ln(b)}\bigg|_{R\to 0} = 0, \tag{125}$$

where the last equality is valid under the (correct) assumption that the renormalization of c is not singular in the replica limit.

Therefore, in the chiral classes, conductivity is not renormalized for symmetry reasons (i.e. independently of the topology encoded in the level k of the WZW term).

We will now find the renormalization group flow of $S[V]$.

Splitting the fields As a first step, we split the fields $V = V_s V_f$ into slow and fast components. We exploit Eq. (112)

$$S[V_s V_f] = S[V_s] + S[V_f] + S_{\text{int}}[V_s, V_f], \tag{126}$$

where

$$S_{\text{int}}[V_s, V_f] = \frac{1}{4\pi}\int_{\mathbf{x}} \text{tr}\,[\underbrace{V_f^\dagger \partial_\mu V_f}_{\simeq [W,\partial_\mu W]/2}\ \underbrace{(\sigma\delta_{\mu\nu} - ik\epsilon_{\mu\nu})V_s^\dagger \partial_\nu V_s}_{\equiv A_\mu^s}]. \tag{127}$$

Note that in the expansion of the fast fields in generators $V_f = e^{iW}$, the leading term is quadratic (the linear term in W would imply an integral $\int_{\mathbf{p}} W(\mathbf{p})A^s_\mu(\mathbf{p})p_\mu$, but slow fields and the fast field $W(\mathbf{p})$ have no common support in momentum space).

It is convenient to expand the fast fields in normalized generators of $SU(R)$, $W = \phi^a t_a$, $\text{tr}\,[t_a t_b] = 2\delta_{ab}$, so that fast propators are

$$\langle \phi_a(\mathbf{p})\phi_b(\mathbf{p}')\rangle = \frac{\delta_{ab}2\pi}{\sigma \mathbf{p}^2}(2\pi)^2\delta(\mathbf{p}+\mathbf{p}'). \tag{128}$$

RG step. Then we find

$$\begin{aligned}
\delta S_s &= -\frac{1}{2}\langle S^2_{\text{int}}\rangle_f \\
&= \frac{1}{32\pi^2}\int_{\mathbf{p},\mathbf{q},\mathbf{p}',\mathbf{q}'} \text{tr}\,[A^s_\mu(\mathbf{q})t^a t^b]\text{tr}\,[A^s_{\mu'}(\mathbf{q}')t^{a'}t^{b'}]p_\mu p'_{\mu'}\langle \phi^a(-\mathbf{p}-\mathbf{q})\phi^b(\mathbf{p})\phi^{a'}(-\mathbf{p}'-\mathbf{q}')\phi^{b'}(\mathbf{p}')\rangle \\
&= \dots \qquad (129)\\
&= \frac{1}{8\pi\sigma^2}\ln(b)R\int_{\mathbf{x}} \text{tr}\,[(A^s_\mu)^2] \qquad (130)
\end{aligned}$$

the ellipsis denotes some straightforward but tedious algebra (see p. 460 of Altland and Simons or ask me for some handwritten notes).

The crucial ingredient of topology enters in

$$\mathrm{tr}\,[(A^s_\mu)^2] = \mathrm{tr}\,[((\sigma\delta_{\mu\nu} - ik\epsilon_{\mu\nu})V_s^\dagger\partial_\nu V_s)^2] = -(\sigma^2 - k^2)\mathrm{tr}\,[\partial_\mu V_s^\dagger \partial_\mu V_s]. \tag{131}$$

From simple differentiation we see that we have thus derived the RG equation for the $SU(R)_k$ WZNW model in class AIII. We present it along with the RG equations for the other two WZNW models

$$\frac{d\sigma}{d\ln(b)} = -\frac{\sigma^2 - k^2}{\sigma^2}R \quad \xrightarrow{R\to 0} 0 \qquad SU(R)_k \quad \text{class AIII} \tag{132}$$

$$\frac{d\sigma}{d\ln(b)} = -\frac{\sigma^2 - k^2}{\sigma^2}(R-2) \xrightarrow{R\to 0} 2\frac{\sigma^2 - k^2}{\sigma^2} \qquad O(R)_k \quad \text{class DIII} \tag{133}$$

$$\frac{d\sigma}{d\ln(b)} = -\frac{\sigma^2 - k^2}{\sigma^2}(R+1) \xrightarrow{R\to 0} -\frac{\sigma^2 - k^2}{\sigma^2} \qquad Sp(2R)_k \quad \text{class CI} \tag{134}$$

$$\tag{135}$$

Recall that for class AIII there is one more coupling constant, which has a flow

$$\frac{dc}{d\ln(b)} = -\frac{1}{R}\frac{d\sigma}{d\ln(b)}\bigg|_{R\to 0} = \frac{\sigma^2 - k^2}{\sigma^2}. \tag{136}$$

Comments

- The RG flow of the non-topological system ($k = 0$) is critical, antilocalizing and localizing in classes AIII, DIII and CI respectively.
- The RG flow of the topological system is fundamentally different:
  - At finite k and finite R (as well as the flow for class CI in the replica limit) it is towards a critical fix point at $\sigma = k$. This encodes the protection of 3D TI surface states. The coefficient $\sigma^2 - k^2$ appears in all loop orders, so it will always protect, see Fig. 3.3.4.
  - For the model of class AIII presented in Eq. (115), we found a starting value $\sigma = k$ using non-Abelian bosonization. Thus we reside at the critical point and c doesn't flow either (this model displays as line of critical points parametrized by λ_A).
- The 1-loop RG equations of class AIII are known to be exact (i.e. all higher loop orders vanish. Topological vortex contributions can appear however).

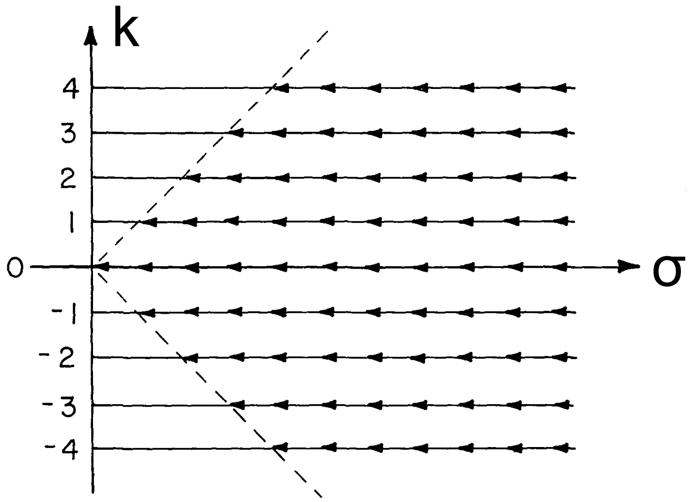


Figure 7: RG flow of WZNW models (adapted from Witten, Commun. Math. Phys. 92, 455 (1984).

3.3.5 Disordered Dirac fermions II: Tractable models of multifractality

So far we have seen

- Anderson Metal-Insulator-transition in standard symmetry classes
- The impact of a θ term leading to bulk topological phases and phase transitions
- The impact of the WZW term which prevents localization.

These calculations substantiate the picture of 1 respectively 2 parameter scaling at Anderson transitions.

Now we will see that actually, at Anderson criticality, there are not 1 or 2 operators with non-trivial scaling dimension, but infinitely many.

Dirac electron with random vector potential - explicit structure of wave function We consider the $k = 1$ version of Eq. (115), i.e. the surface state of an 3D topological insulator with bulk winding $k = 1$ (i.e. with one Dirac electron on its surface.

The Hamiltonian is

$$H = v(-i\partial_\mu + A_\mu)\sigma_\mu \tag{137}$$

and set $v = 1$ and

$$\langle A_\mu(\mathbf{x})A_\nu(\mathbf{x}')\rangle_{\text{dis.}} = \delta_{\mu\nu}\lambda_A\delta(\mathbf{x} - \mathbf{x}'). \tag{138}$$

Since $\{H, \sigma_z\} = 0$, the model has chiral symmetry and no other symmetries and belongs to class AIII. It describes the surface state of some exotic topological insulator.

We use the notation $A_\mu = \epsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\chi$, gauge away χ realize that Eq. (138) implies

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle_{\text{dis.}} = \delta_{\mu\nu}\frac{\lambda_A}{2\pi}\ln(L/|\mathbf{x}-\mathbf{x}'|) \tag{139}$$

and obtain zero modes

$$\psi_\pm = \mathcal{N}_\pm[\phi]e^{-\phi(\mathbf{x})\sigma_z}\hat{e}_\pm \tag{140}$$

where $\hat{e}_+ = (1,0), \hat{e}_- = (0,1)$. Clearly, $\mathcal{N}_\pm[\phi]^2 = \int d^2x e^{\mp 2\phi(\mathbf{x})}$.

Are these exact solutions localized or insulating? The answer is neither nor.

Generalities on Multifractality As a measure of localization, we define the inverse participation ratio

$$\mathcal{P}_q = \int d^dx|\psi(\mathbf{x})|^{2q} \tag{141}$$

and study its scaling with system size

$$\mathcal{P}_q \sim L^{-\tau_q} \tag{142}$$

Note that per definition $\tau_0 = -d$ and $\tau_1 = 0$ (normalization.) We readily distinguish different cases

- $\tau_q = 0 \forall q > 0$: Insulator
- $\tau_q = d(q-1)$: Metal
- Intermediate cases: *Multifractal*

Multifractality spectrum of Dirac fermions in random vector potential In the present case

$$\mathcal{P}_q = \langle\frac{\int d^2xe^{\mp 2q\phi}}{[\int d^2xe^{\mp 2\phi}]^q}\rangle_{\text{dis.}} \tag{143}$$

The general calculation is a bit cumbersome, but for weak disorder, $\lambda_A < 2\pi$, it turns out that it is sufficient to average numerator and denominator separately

$$\mathcal{P}_q \simeq \frac{\langle\int d^2xe^{\mp 2q\phi}\rangle_{\text{dis.}}}{[\langle\int d^2xe^{\mp 2\phi}\rangle_{\text{dis.}}]^q} \tag{144}$$

We use

$$\langle \int d^2x e^{\mp 2q\phi} \rangle_{\text{dis.}} \sim (L/a)^2 \langle e^{\mp 2q\phi} \rangle_{\text{dis.}} \sim (L/a)^{2-(2q)^2\lambda_A/(4\pi)} \tag{145}$$

so that in total

$$\mathcal{P}_q \sim \frac{(L/a)^{2-q^2\lambda_A/\pi}}{[(L/a)^{2-\lambda_A/\pi}]^q} \sim (L/a)^{-(q-1)(2-\lambda_A q/\pi)} \tag{146}$$

Notes

- We see that $\tau_q = (q-1)(2-\lambda_A q/\pi)$ is non-trivial, so the wave function is multifractal.
- Note that the scaling dimension continuously changes as a function of λ_A, i.e. we have a line of fixed points
- Recall that this is valid only for $\lambda_A < 2\pi$, at larger λ_A the spectrum is frozen (i.e. $\tau_q \equiv 0$ for all $q > q_c$ and $0 < q_c < 1$.)

Part II

Many-Body physics in the presence of disorder

In this part we will consider many-body problems in the presence of disorder.

4 Disordered electronic problems with interactions

(This section is largely copy-pasted from an introductory chapter in my PhD thesis.)

4.1 Motivation

We have seen

- Single particle phenomenon of Anderson localization
- Single particle (eigenstate) quantum phase transition and criticality
- precursor of strong Anderson localization: Weak localization corrections.

We have also discussed that electron-electon interactions cut the weak localization corrections (or the RG-flow) and serve as an IR cut-off, see discussion around Eq. (69).

However, there are additional effects of interactions

- Additional localizing effects to conductivity (Altshuler-Aronov corrections, $\delta\sigma = -\frac{1}{\pi}\ln(1/T\tau)$, see Fig.8)
- Zero bias anomaly in the tunneling density of states
- Antagonistic interplay Superconductivity vs. Anderson localization (Anderson-theorem)

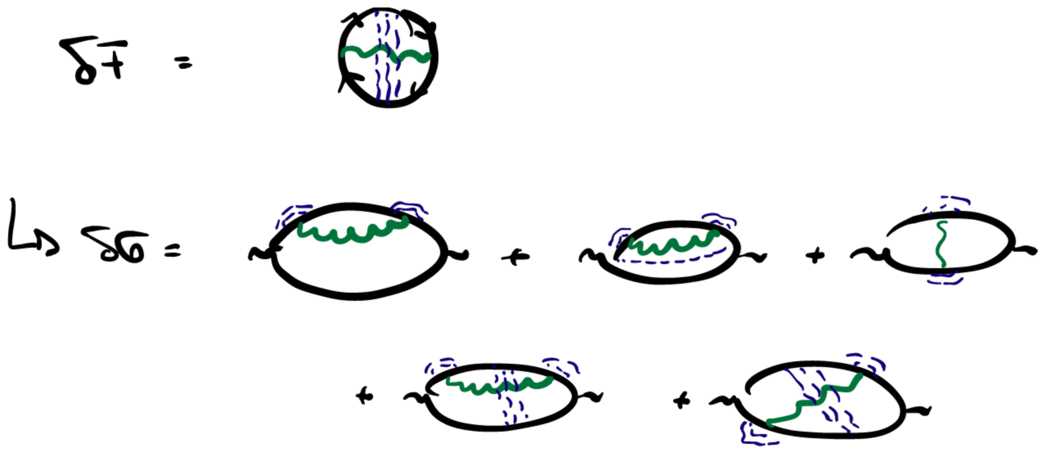


Figure 8: Altshuler Aronov corrections to the conductivity: The 5 diagrams associated to this correction can be obtained from the leading order Free energy by diagramatic insertion of conductivity vertices (i.e. differentiation)

4.2 Clean and disordered Fermi liquid

The main statement of Landau's Fermi liquid theory [Nozieres and Luttinger(1962), Abrikosov *et al.*(196 Landau *et al.*(1980)Landau, Lifshits, and Pitaevskii] is that, in the absence of spontaneous symmetry breaking,[2] the low-energy[3] excitations of a strongly correlated fermionic system are fermions (quasiparticles) with the same quantum number as the free particles. Their decay rate is small compared to the Fermi energy.

Landau's phenomenological theory can be put on firm ground using field theoretic techniques. The exact electronic Green's function (i.e. two point correlator) can be shown to contain a singular part (quasiparticle pole) with weight $0 < a < 1$ and an additional regular contribution [Luttinger(1961)]. In the entire lecture notes I reabsorb a into a redefinition of fermionic fields and scattering amplitude.

Another particularly important quantity in the QFT of strongly interacting fermions are the four point correlators. These implicitly define the full interaction amplitudes. The latter are subdivided in different channels of small energy-momentum transfer according to their tensor structure in spin space, see Fig. 9. They include, among others, resonant particle-hole (particle-particle) bubbles, a singular contribution when the two fermionic propagators have close d+1 momentum. In order to handle the divergence, the scattering amplitude I is defined (in each channel): It contains only the one bubble irreducible diagrams of Γ and is thus regular. By definition its resummation with bubbles equals the full Γ (for simplicity any spin structure is omitted here).

[2] The spectrum of the interacting system is adiabatically connected to the free problem.

[3] Low energy as compared to the Fermi energy.

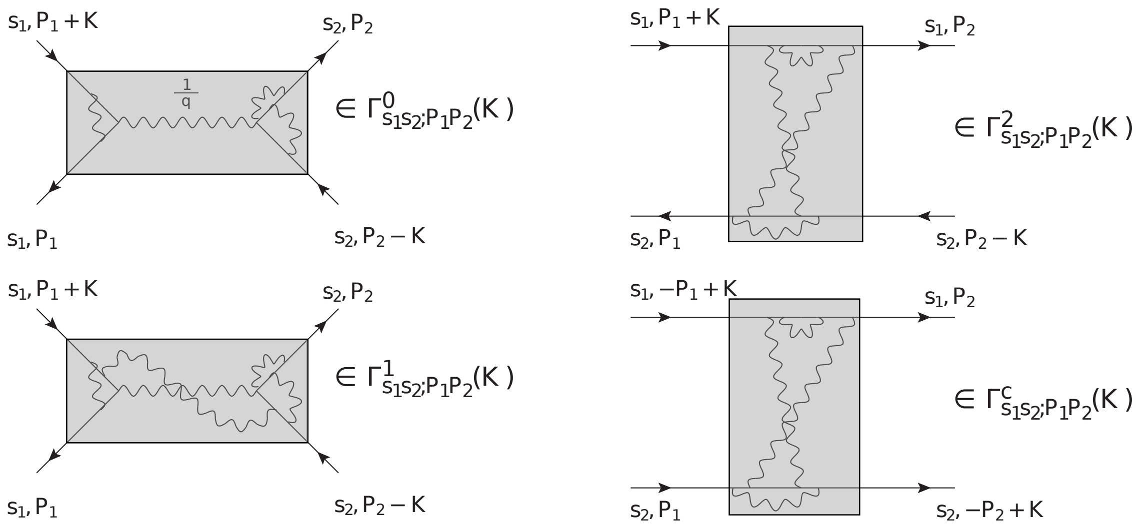
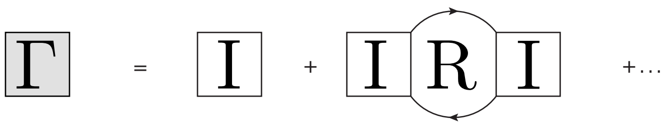


Figure 9: Interaction channels of small energy momentum transfer (capital letters P, K) in the Fermi Liquid theory. Classification according to the tensor structure in spin-space (index s_i). Upper left: Exemplary contribution to the one Coulomb-line reducible small angle scattering amplitude. Lower left: Exemplary contribution to the one-Coulomb-line irreducible small-angle scattering amplitude. Upper right: Exemplary contribution to the large-angle scattering amplitude. Lower right: Exemplary contribution to the scattering amplitude in the Cooper channel.



The bubble of a particle and a hole (or particle) of similar d+1 momenta has a singular and a regular contribution

$$R_{n,\mathbf{p}}\left(\omega_m,\mathbf{q}\right)\equiv G\left(\epsilon_{n_i},\mathbf{p}_i\right)G\left(\epsilon_{n_i}+\omega_m,\mathbf{p}_i+\mathbf{q}\right)=-\beta\frac{v_F\hat{p}\cdot\mathbf{q}}{v_F\hat{p}\cdot\mathbf{q}-i\omega_n}\delta_P+reg. \tag{147}$$

at vanishing $(\omega_m,\mathbf{q})$. (δ_P constrains the fast energy-momentum P on the Fermi surface.)

From the above expressions it becomes evident that the ω-limit of the scattering amplitude[4]

$$\Gamma^{\omega}=\lim_{\omega\to 0}\left(\lim_{q\to 0}\Gamma\left(\omega,\mathbf{q}\right)\right) \tag{148}$$

[4]The limiting procedure of Matsubara frequencies involves analytic continuation and is explained in Ref. [Nozieres and Luttinger(1962)].

is regular at vanishing $(\omega_m, \mathbf{q})$ and the dependence of $\Gamma_{\hat{p},\hat{p}'}(\omega_m, \mathbf{q})$ on the transferred $d+1$ momentum is governed by the multiple resummation

$$\Gamma_{\hat{p},\hat{p}'}(\omega_m, \mathbf{q}) = \Gamma^{\omega}_{\hat{p},\hat{p}'} + \int_{\hat{p}_i} \Gamma^{\omega}_{\hat{p},\hat{p}_i} \frac{-v_F \hat{p}_i \cdot \mathbf{q}}{v_F \hat{p}_i \cdot \mathbf{q} - i\omega_m} \Gamma^{\omega}_{\hat{p}_i,\hat{p}'} + \dots \ . \tag{149}$$

It can be shown that $\Gamma^\omega = F$ where F are the phenomenological Fermi liquid interaction parameters.

As will be explained below, for the purpose of disordered systems it is more useful to extract the static limit

$$\Gamma^q = \lim_{q\to 0}\left(\lim_{\omega\to 0}\Gamma(\omega, \mathbf{q})\right) \tag{150}$$

of $\Gamma(\omega, \mathbf{q})$ (i.e. F summed up with retarded-retarded/advanced-advanced bubbles). Using this quantity the full amplitude is given by

$$\Gamma_{\hat{p},\hat{p}'}(\omega_m, \mathbf{q}) = \Gamma^{q}_{\hat{p},\hat{p}'} + \int_{\hat{p}_i} \Gamma^{q}_{\hat{p},\hat{p}_i} \frac{-i\omega_m}{v_F \hat{p}_i \cdot \mathbf{q} - i\omega_m} \Gamma^{q}_{\hat{p}_i,\hat{p}'} + \dots \ . \tag{151}$$

All presented formulas apply to the short range interactions Γ^1, Γ^2 and Γ^c. The long-range interaction Γ^0 requires special treatment. It needs to be RPA screened by the full Fermi liquid polarization operator $\Pi(\omega_m, \mathbf{q})$ which involves short range interactions, see Fig. 11.

Upon inclusion of sufficiently weak disorder (in the sense $k_f\ell \gg 1$) the q-limit quantities (e.g. Γ^q) remain unchanged, because they are determined by scales much shorter than the mean free path. On the contrary, the dynamic part of scattering amplitudes, i.e. the retarded-advanced particle-hole/particle-particle bubbles, has to be replaced by its diffusive counterpart [Finkelstein(1990), Finkel'stein(2010)]:

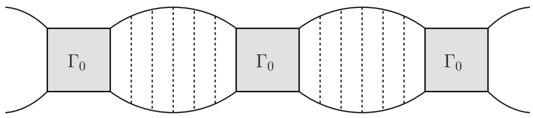
$$\frac{\omega_n}{\omega_n + i v_F \hat{p}\cdot \mathbf{q}} \to \frac{\omega_n}{\omega_n + Dq^2} \tag{152}$$

This replacement is best understood diagrammatically: All channels $\Gamma(\omega, \mathbf{q})$ are determined by infinite resummation of Γ^q with retarded-advanced particle-hole (particle-particle) bubbles. In the disordered case the latter become Diffusons (Cooperons), see Fig. 10 for the exemplary case of Γ^0.

Another important consequence of disorder is to gap out all harmonics of the particle-hole bubbles except the zeroth harmonic (broken rotational symmetry). Therefore, the higher harmonics of $\Gamma_{\hat{p}_1,\hat{p}_2}$ are not coupled to the long-range diffusive modes and I henceforth denote $\Gamma = \langle \Gamma_{\hat{p}_1,\hat{p}_2}\rangle_{\text{Fermi surface}}$.

4.3 Non-linear sigma model with interactions

On the basis of the disordered Fermi liquid, the diffusive NLSM of interacting electrons can be derived [A.M.Finkel'stein(1983), A.M.Finkel'stein(1984)]. As before, the QFT can be



$$= \Gamma^{0,q} \tfrac{\omega_n}{\omega_n + Dq^2} \Gamma^{0,q} \tfrac{\omega_n}{\omega_n + Dq^2} \Gamma^{0,q}$$

Figure 10: A diagram contributing in the resummation of Γ^0 in the diffusive limit. The retarded-advanced sections become diffusons for energies below $1/\tau$.

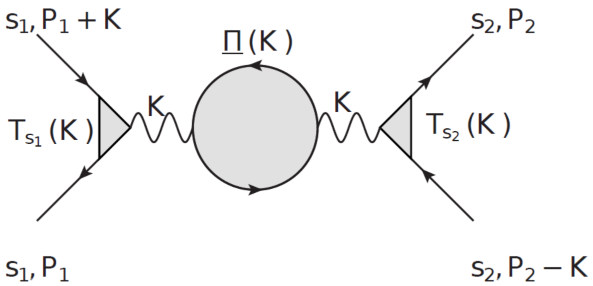


Figure 11: The long range part of the Coulomb interaction has to be screened by means of the full polarization operator and vertex corrections have to be included.

obtained by functional integration and gradient expansion, again it is constructed on the saddle point manifold of the *free* problem. The interaction terms then formally correspond to particular mass terms in the action:

$$S = S_\sigma + S_{int}^{(\rho)} + S_{int}^{(\sigma)} + S_{int}^{(c)}, \tag{153a}$$

with

$$S_\sigma = \frac{\sigma}{32} \int_{\mathbf{x}} \text{tr} \left[(\nabla Q)^2 \right] - 4\pi T z \int_{\mathbf{x}} \text{tr}\, \eta Q, \tag{153b}$$

$$S_{int}^{(\rho)} = \frac{\pi T}{4} \Gamma_\rho \sum_{\alpha,n} \sum_{r=0,3} \int_{\mathbf{x}} \text{tr} \left[I_n^\alpha t_{r0} Q \right] \text{tr} \left[I_{-n}^\alpha t_{r0} Q \right], \tag{153c}$$

$$S_{int}^{(\sigma)} = \frac{\pi T}{4} \Gamma_t \sum_{\alpha,n} \sum_{r=0,3} \sum_{j=1,2,3} \int_{\mathbf{x}} \text{tr} \left[I_n^\alpha t_{rj} Q \right] \text{tr} \left[I_{-n}^\alpha t_{rj} Q \right], \tag{153d}$$

$$S_{int}^{(c)} = \frac{\pi T}{2} \Gamma_c \sum_{\alpha,n} \sum_{r=0,3} (-)^r \int_{\mathbf{x}} \text{tr} \left[I_n^\alpha t_{r0} Q I_n^\alpha t_{r0} Q \right]. \tag{153e}$$

In this case, where both time reversal and spin-rotational invariance are assumed (non-interacting class AI), the Q matrices are symplectic, traceless and involutive and have non-trivial structure in replica, Matsubara, spin and Nambu spaces. Following Ref. [Burmistrov *et al.*(2012)B I here use the convention $t_{rj} = \tau_r \otimes \sigma_j$ where $\tau_r = (\mathbf{1}_\tau, \vec{\tau})$ are the identity and the Pauli matrices in Nambu space, while $\sigma_j = (\mathbf{1}_\sigma, \vec{\sigma})$ are those in spin space. The following matrices,

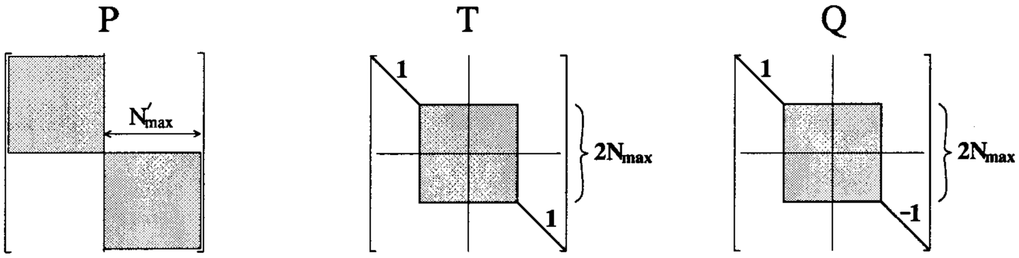


Figure 12: In order to ensure gauge invariance w.r.t. rotations which are local in time (electrostatic gauge invariance = Coulomb interactions), a two cut-off scheme can be used [Pruisken and Baranov(1995)]: Q and T matrices are only $N_M \times N_M$, while the space where gauge transformations act (e.g. where gauge potetials live) is $N'_M \times N'_M$ with $N'_M \gg N_M$ (see the matrix denoted P.

which are trivial in Nambu and spin spaces, occur:

$$\begin{aligned} \Lambda^{\alpha\beta}_{nm} &= \mathrm{sgn}\,(n)\,\delta^{\alpha\beta}\delta_{nm}, \\ \eta^{\alpha\beta}_{nm} &= n\delta^{\alpha\beta}\delta_{nm}, \\ \left(I^{\alpha_0}_{n_0}\right)^{\alpha\beta}_{nm} &= \delta^{\alpha_0\alpha}\delta^{\alpha_0\beta}\delta_{n-m,n_0}. \end{aligned} \tag{154}$$

The coupling constants of the NLSM are the dimensionless conductivity σ, the $q-$limits of interaction amplitudes in density (singlet), triplet and Cooper channel

$$\Gamma_\rho = -\frac{\pi\nu}{4}\left(2\Gamma^{0,q} + 2\Gamma^{1,q} - \Gamma^{2,q}\right), \tag{155a}$$

$$\Gamma_t = \frac{\pi\nu}{4}\Gamma^{2,q}, \tag{155b}$$

$$\Gamma_c = \frac{\pi\nu}{4}\Gamma^{c,q}, \tag{155c}$$

and the prefactor z of the frequency term, which is related to the renormalization of specific heat. Note that it does not flow in the non-interacting case and keeps the bare value $z^{(0)} = \pi\nu/4$. In the presence of long range Coulomb interaction ($\Gamma^0 \neq 0$) the NLSM is "$\mathcal{F}$-invariant" [Pruisken *et al.*(1999)Pruisken, Baranov, and Skoric]. Essentially this means electrostatic gauge invariance (i.e. invariance under time dependent but space independent phase rotations) and fixes $z + \Gamma_\rho = 0$. To practically incorporate such gauge transformations, a two cut-off scheme may be imposed, see Fig. 12.

The relationship $z + \Gamma_\rho = 0$ is readily clear in the limit of weak interactions, where RPA works and $U_{\mathrm{RPA}}(\mathbf{q}) = U(\mathbf{q})/[1 + U(\mathbf{q})\nu] \stackrel{\mathbf{q}\to 0}{\to} 1/(2\nu)$, such that $\Gamma^{0,q} = 1/2$ [recall that ν is the DOS per spin].

The one loop RG equations are [Finkelstein(1990), Belitz and Kirkpatrick(1994), Burmistrov *et al.*(2012

Dell'Anna(2013)]

$$
\begin{aligned}
\frac{1}{t}\frac{dt}{d\ln(b)} &= t\left[1 + f(\gamma_\rho) + 3f(\gamma_t) - \gamma_c\right], & \text{(156a)}\\
\frac{d\gamma_\rho}{d\ln(b)} &= -\frac{t}{2}\left(1+\gamma_\rho\right)\left(\gamma_\rho + 3\gamma_t + 2\gamma_c\right), & \text{(156b)}\\
\frac{d\gamma_t}{d\ln(b)} &= -\frac{t}{2}\left(1+\gamma_t\right)\left(\gamma_\rho - \gamma_t - 2\gamma_c\left(1+2\gamma_t\right)\right), & \text{(156c)}\\
\frac{d\gamma_c}{d\ln(b)} &= -\frac{t}{2}\left[\left(1+\gamma_c\right)\left(\gamma_\rho - 3\gamma_t\right) + 6\gamma_c\left(\gamma_t - \ln\left(1+\gamma_t\right)\right)\right] - 2\gamma_c^2. & \text{(156d)}\\
\frac{d\ln(z)}{d\ln(b)} &= \frac{t}{2}(\gamma_s + 3\gamma_t + 2\gamma_c) & \text{(156e)}
\end{aligned}
$$

(The running scale is $y = \ln L/l$.) It is a necessary consequence of dimensional analysis, that the RG equations can be written in terms of reduced coupling constants $\gamma_i = \frac{\Gamma_i}{z}$ $(i = \rho, t, c)$. The function

$$
f(x) = 1 - \frac{1+x}{x}\ln(1+x) \sim \begin{cases} 1, & \text{as } x \to -1, \\ -\frac{x}{2}, & \text{as } x \to 0, \end{cases} \tag{157}
$$

was introduced. These RG equations are perturbative in $t = 2/(\pi\sigma)$ and γ_c but exact in γ_ρ and γ_t. I neglected terms beyond leading order in the small parameters t, $t\gamma_c$ on the RHS of Eqs. (156).

Eqs. (156) contain the WL effect [first term "1" in the square bracket of Eq. (156a)], which is also present in non-interacting systems and the Cooper instability, last term "$-2\gamma_c^2$" in Eq. (156d) which is also present in clean systems [Shankar(1994)]. All other terms stem from the interplay of disorder and interactions, in particular the second term "$f(\gamma_\rho)$" in the square bracket of Eq. (156a) reproduces the AA effect. Note the preservation of $\mathcal{F}$-invariance at $\gamma_\rho = -1$.

In the case of a system with strong spin-orbit coupling, i.e. class AII (for example for 3D topological insulator surface states) the following modifications to Eqs. (156) occur. First, one should replace the Weak localization by the WAL effect, i.e. "1" in the square bracket of Eq. (156a) by "$-1/2$". Second, the triplet channel is gapped out and γ_t should be removed from all equations.

In the following, I would like to briefly expose some outstanding scientific questions which can be solved by means of the interacting NLSM.

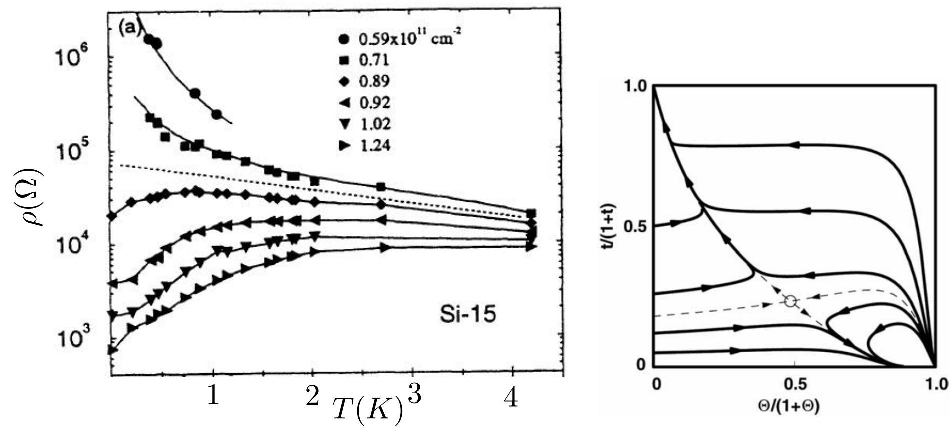


Figure 13: Left: Temperature dependent resistivity for various carrier densities [Kravchenko *et al.*(1994)Kravchenko, Kravchenko, Furneaux, Pudalov, and D'Iorio]. Note the non-monotonic curve below the dotted separatrix. Right: 2 parameter scaling proposed by Punnoose and Finkelstein (Eqs. (156) in the limit of large valley number $n_{\text{valley}}t \to t$, $n_{\text{valley}}\gamma_t \to \Theta$), so that the Stoner instability at finite T is switched off artificially)

4.4 Metal-insulator transition

In contrast to the expectation founded on the scaling theory (Sec. **??**), Kravchenko *et al.* [Kravchenko *et al.*(1994)Kravchenko, Kravchenko, Furneaux, Pudalov, and D'Iorio] experimentally discovered a Metal Insulator Transition in the 2D electrong gas created in Silicon field effect transistors (see Fig. 13). The effect crucially depends on the (relatively small) carrier concentration[5] $n \sim 10^{11}cm^{-2}$ while the sensitivity to an in-plane magnetic field hints to the importance of the electronic spin. These curious findings were subsequently theoretically explained by Punnoose and Finkelstein [Punnoose and Finkel'stein(2005)] in an analysis based on Eqs. (156) for the case of Coulomb interaction $\gamma_\rho = -1$ and Cooper repulsion $\gamma_c > 0$ (then $\gamma_c \to \sqrt{t} \approx 0$ very quickly under RG).

According to the RG-equations, the resistance initially increases at small γ_t (insulating behavior due to Altshuler-Aronov and Weak localization effects). However, $\gamma_t > 0$ increases itself under RG (an effect missed in perturbation theory) and eventually the antilocalizing "$3f(\gamma_t)$"-term in Eq. (156b) dominates. This leads to non-monotonic (and eventually metallic) resistivity curves as in Fig. 13) and proves the presence of the delocalized phase. On the other hand at sufficiently strong disorder the system has to be insulating (Anderson localization) and therefore a Metal Insulator Transition in between of the two phases has to exist.

[5] Recall, that the interaction strength (determined by the dimensionless density parameter r_s) is strong for dilute and weak for dense concentrations respectively.

The above theory has two important drawbacks: First, as usual, being perturbative in t it can not treat the region of the actual phase transition at the order of $t \sim 1$. Second, and more severely, the above RG-equations imply $\gamma_t \to \infty$ (corresponding to strong spin correlations) at some finite temperature and the theory breaks down. A way out of this dilemma is provided by a two-loop calculation combined with expansion in large valley number $n_V \gg 1$ [Punnoose and Finkel'stein(2005)].

4.5 Superconducting transition in amorphous films

Next, I would like to review the application of the interacting NLSM to the superconducting transition in disordered materials (I thus consider $\gamma_c < 0$ in this section). Before doing so, it is beneficial to place the NLSM treatment in the context of various different theories concerning this problem.

4.5.1 A quick review

Superconductivity, i.e., the phenomenon of frictionless transport and perfect diamagnetism, is a consequence of long-ranged correlations of the complex order parameter $\Delta(\mathbf{x})$ in a theory of charged particles:

$$\langle \Delta(\mathbf{x})\Delta(0)\rangle \overset{x\to\infty}{\sim} \begin{cases} e^{-x/\xi} & \text{exponential decay in the normal state } (\xi \text{ is the correlation length}), \\ \langle\Delta\rangle^2 & \text{constant in the superconducting state.} \end{cases} \tag{158}$$

In two spatial dimensions at finite temperature[6] true long-range order is not possible in view of the Mermin-Wagner theorem. In this case, one resorts to the following weaker definition:

$$\langle \Delta(\mathbf{x})\Delta(0)\rangle \overset{x\to\infty}{\sim} \begin{cases} e^{-x/\xi} & \text{exponential decay in the normal state,} \\ 1/x^\eta & \text{algebraic decay } (0<\eta<1) \text{ in the superconducting state.} \end{cases} \tag{159}$$

Typically, the following two sufficient conditions are fulfilled in a superconductor:

1. The modulus of the expectation value $|\langle\Delta(\mathbf{x})\rangle|$ is non-vanishing and (nearly) homogeneous.
2. Strong phase fluctuations of $\phi = \arg\left(\langle\Delta(\mathbf{x})\rangle\right)$ are suppressed due to sufficiently large phase rigidity.

As a consequence of these conditions, two different mechanisms driving the transition between the superconducting and the normal state are often distinguished [Gantmakher and Dolgopolov(2

[6]... or in one spatial dimension at zero temperature...

1. The expectation value $\langle \Delta(\mathbf{x}) \rangle$ vanishes across the transition.
2. The expectation value $\langle \Delta(\mathbf{x}) \rangle \neq 0$ is locally finite, but the phase rigidity vanishes across the transition.

The first of these two mechanisms is sometimes referred to as "fermionic" scenario. It includes the Bardeen-Cooper-Schrieffer theory and related theories, in particular also the NLSM treatment. In contrast, in the second "bosonic" mechanism, the phase fluctuations of preformed Cooper pairs drive the transition, typically the fermionic spectrum displays a pseudogap even in the normal state.

A particularly important representative of bosonic theories is the Berezinskii Kosterlitz Thouless theory. In the following I will return to the fermionic mechanism and the review of RG in the interacting, diffusive NLSM.

4.5.2 Suppression of T_c in the presence of Coulomb interaction

The transition temperature T_c of superconductivity corresponds to the running scale y_c at which the Cooper channel coupling constant γ_c diverges. Here, I focus on the case of Coulomb interaction ($\gamma_\rho = -1$) and negligible triplet channel $\gamma_t \approx 0$ [Finkel'stein(1987), Finkelstein(1990), Finkel'stein(1994)]. When T_c does not differ too much from the clean mean field temperature T_{BCS} it is justified to solve Eq. (156d) at given fixed t. The solution

$$\gamma_c(y) = \frac{t}{8} + \frac{\sqrt{\frac{t}{4} + \left(\frac{t}{8}\right)^2}}{\tanh\left[\sqrt{\frac{t}{4} + \left(\frac{t}{8}\right)^2} 2(y - y_c)\right]} \tag{160}$$

has to obey the boundary condition $\gamma_c(0) \equiv \gamma_c^{(0)} = \frac{1}{\ln T_{BCS}\tau}$ leading to

$$\frac{T_c}{T_{BCS}} = e^{-\frac{1}{\gamma_c^{(0)}}} \left(\frac{1 + \frac{\sqrt{t}/2}{\gamma_c^{(0)} - t/8}}{1 - \frac{\sqrt{t}/2}{\gamma_c^{(0)} - t/8}} \right)^{\frac{1}{\sqrt{t}}} \tag{161}$$

and thus to suppression of T_c. This formula is in nice agreement with experimental data [Finkel'stein(1987)].

4.5.3 Enhancement of T_c in the presence of short-range interaction

Equally, it is very instructive to consider the limit of short-range interaction and strong disorder for Eqs. (156) [Burmistrov *et al.*(2012)Burmistrov, Gornyi, and Mirlin]. Then

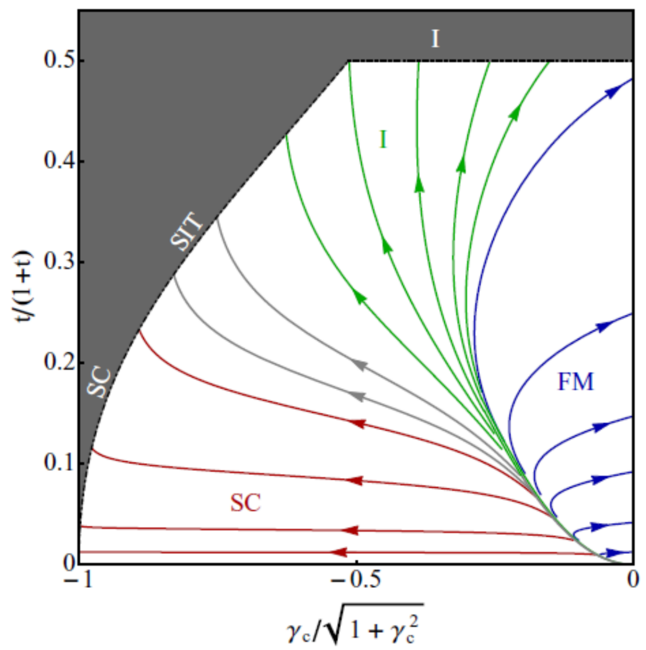


FIG. 1. (Color online) The case of preserved spin-rotational symmetry with the Coulomb interaction, $\gamma_s = -1$: the RG flow obtained from a numerical solution of Eqs. (30)–(32). The initial condition fixes $\gamma_{t0} = 0.2$. The arrows indicate the flow towards the infrared. The gray region indicates the part of the phase diagram that is not accessible within one-loop RG equations. The lines describing the flow to the superconducting (SC), insulating (I), and ferromagnetic (F) phases are shown in red, green, and blue, correspondingly. Gray flow lines correspond to the region of superconductor-insulator transition (SIT).

Figure 14: RG flow in the limit when $\gamma_\rho = -1$. Taken from Burmistrov et al. PHYSICAL REVIEW B 92, 014506 (2015).

$|\gamma_i^{(0)}| \equiv \gamma_i(0) \ll t^{(0)} \equiv t(0) \ll 1$ at the UV scale and all interaction corrections to the resistance can be neglected. Similarly, Eqs. (156b)-(156d) can be linearized keeping as the only non-linearity the clean term $-2\gamma_c$ (Cooper instability). It is smaller than the linear terms, and therefore omitted in the first part of a two-step RG. In this first step the system quickly (at RG scale $y \sim 1$) adjusts to $-\gamma_\rho = \gamma_t = \gamma_c$. In the subsequent second RG step, these couplings suffice

$$\frac{d\gamma_c}{dy} = 2t\gamma_c - 2\gamma_c^2/3. \tag{162}$$

Now there is a trade-off: when $\gamma_i^{(0)} \ll (t^{(0)})^2 \; \forall i = s, t, c$ the system flows to an insulator before any instability can occur. Contrary, if $(t^{(0)})^2, |\gamma_{s,t}| \ll |\gamma_c| \ll t^{(0)}$ the instability occurs before localization. Again the scale of divergence dictates the transition temperature

$$\frac{T_c}{T_{BCS}} \sim e^{-\frac{1}{\gamma_c^{(0)}}} e^{-\frac{2}{t^{(0)}}\left(1-\frac{t^{(0)}}{t(y_c)}\right)}. \tag{163}$$

Note that, because of the condition $-\gamma_c^{(0)} \ll t^{(0)}$ the first exponential dominates and leads to enhancement for T_c. The physical mechanism behind this phenomenon is wavefunction multifractality, which leads to an enhancement of matrix elements of interaction.

Part III

Quantum Chaos - single particle vs many particle

5 Motivation - Random matrix theory and finite samples of diffusive metals

Consider a diffusive piece of metal with size larger than mean free path but smaller than the localization length $\ell < L < \xi$. The kinetic term in the NLσM, Eq. (31), leads to a quantization of modes in the propagator $(q^2 - i\omega/D)^{-1}$, with $\mathbf{q} = 2\pi\mathbf{n}/L$, $\mathbf{n} \in \mathbb{Z}^d$. At energies below the *Thouless energy*, $1/\tau_{\text{Th}} = D/L^2$, all modes except $n = 0$ are gapped.

We thus found a universal action, independent of system geometry and/or microscopic details such as scattering length

$$S = i\frac{\omega}{2}\text{tr}\,[\Lambda Q] \tag{164}$$

There is only one reference energy left, $\Delta/\pi = L^d\nu$, which is the inverse mean level spacing and used as the fundamental unit of energy.

For simplicity, we consider class A, in which $Q = T^{-1}\Lambda T \in U(2R)/[U(R)\times U(R)]$ and will determine the statistics of eigenvalues, as evidenced from level correlations

$$R_2(\omega) = \Delta^2\langle\rho(\epsilon+\omega/2)\rho(\epsilon-\omega/2)\rangle_{\text{dis}} - 1, \tag{165}$$

where $\rho(\epsilon) = (1/\pi)\text{Im}\,\text{tr}\,[G^A(\epsilon)]$ and tr runs over all internal quantum numbers. We can obtain this from the replica partition function as

$$R_2(\omega) = \frac{1}{2}\left(-\text{Re}\,\left[\lim_{R\to 0}\frac{1}{R^2}\frac{\partial^2}{\partial\omega^2}Z(\omega)\right] - 1\right). \tag{166}$$

We derive the partition sum in the large ω limit .

- First, we determine the saddle point equation

$$[\Lambda, Q] = 0, \tag{167}$$

which is solved by all block diagonal matrices $Q = \text{diag}(Q^R, Q^A)$.

- Solutions can actually be categorized in different sectors
  - The sector with minimal (absolute value) of the action is given by

$$Q_0 = \Lambda = (\underbrace{\ldots 1, 1, 1}_{\text{retarded}}, \underbrace{-1, -1, -1, \ldots}_{\text{advanced}}) \tag{168}$$

  - The second lowest sector (in terms of absolute value of the action) is given by

$$Q_1 = (\underbrace{\ldots 1, 1, -1}_{\text{retarded}}, \underbrace{1, -1, -1, \ldots}_{\text{advanced}}) \tag{169}$$

    and smooth rotations thereof. This is a replica symmetry breaking solution and a saddle (not a minimum) of the $iS[Q]$.
  - However we drop configurations with more than 1 anticausal set of eigenvalues, e.g.,

$$Q_2 = (\underbrace{\ldots 1, -1, -1}_{\text{retarded}}, \underbrace{1, 1, -1, \ldots}_{\text{advanced}}) \tag{170}$$

    as their contribution to the only observable we're considering vanishes in the replica limit.

- The only contributions we keep are thus

$$Z(\omega) = Z_0(\omega) + Z_1(\omega) \tag{171}$$

  - $Z_0(\omega)$: We may expand about the trivial fix point, as in Eq. (37), using $T = e^{W/2}$

$$Q \simeq \Lambda + \Lambda W + \Lambda W^2/2, W = \begin{pmatrix} 0 & w \\ -w^\dagger & 0 \end{pmatrix} \tag{172}$$

    , where in class A w is an arbitrary complex $R \times R$ matrix. The action near this saddle point is

$$S_0[w] = i\omega R - i\omega \mathrm{tr}\,[ww^\dagger]. \tag{173}$$

    Note that there are $2R^2$ real massive modes, i.e.

$$Z_0(\omega) = (i\pi/\omega)^{R^2} e^{-i\omega R} \tag{174}$$

  - $Z_1(\omega)$: Fluctuations about the non-trivial solution imply

$$\begin{aligned} S_1 &= i\frac{\omega}{2}\mathrm{tr}\,[\Lambda T^{-1} Q_1 T] \\ &= i\frac{\omega}{2}\mathrm{tr}\,[Q_1 \Lambda [1 + W^2/2]] \qquad (175) \\ &= i\omega(R-2) - i\omega \mathrm{tr}\,[Q_1 \Lambda \begin{pmatrix} ww^\dagger & 0 \\ 0 & w^\dagger w \end{pmatrix}] \qquad (176) \end{aligned}$$

Note that for the non-trivial saddle point, there are zero modes which are massless and correspond to all possible orientation of placing the causality defying -1 into the retarder sector, and span the manifold

$$\frac{U(R)}{U(1)\times U(R-1)} = CP^{R-1}, \tag{177}$$

in each of retarded and advanced sector. The dimension of this space is $2\times(2R-2)$, so there are $2R^2-2(2R-2) = 2[(R-1)^2+1]$ massive modes, however $2(R-1)^2$ of them have a propagator $1/(-i\omega)$ and two $1/(i\omega)$. The massless modes have to be taken into account exactly (not only on the Gaussian level) and lead to a prefactor

$$Vol\left(\frac{U(R)}{U(1)\times U(R-1)}\right)^2 = \left(\frac{(2\pi)^{R-1}}{\Gamma(R)}\right)^2. \tag{178}$$

Thus the contribution to the partition sum from the first non-trivial sector is

$$Z_1(\omega) = e^{-i\omega(R-2)}\left(\frac{(2\pi)^{R-1}}{\Gamma(R)}\right)^2 (i\pi/\omega)^{(R-1)^2}(-i\pi/\omega) \tag{179}$$

– At any finite R, $Z_{R-1}(\omega) = Z_1(-\omega)$, so we seem to have to keep this sector. However, this is wrong, instead we drop all contributions with with $2 \le p$ anticausal entries in the saddle point solution, see Yurkvich and Lerner, PRB **60** 3958 (1999),
– This leads to

$$Z(\omega) \overset{R\to 0}{\simeq} \frac{e^{-i\omega R}}{\omega^{R^2}} + R^2\frac{e^{i\omega(2-R)}}{4\omega^{(R-1)^2+1}}, \tag{180}$$

where the first term stems from Z_0 and the second from Z_1. From this and Eq. (166) we get

$$\boxed{R_2(\omega) = -\frac{\sin^2(\omega)}{\omega^2}.} \tag{181}$$

Comments

- The final result, Eq. (181) was calculated at large $\omega \gg 1$, but is actually exact.
- There are two main features, see Fig. 15:
  – First, note that levels repel each other: If there is a level at ϵ, it's unlikely to have a level at $\epsilon+\omega$ (this can be understood with a simple Landau-Zener picture) .
  – Second, note the oscillations with frequency $\omega \sim 1/\Delta$, which represents the discreteness of levels in each configuration.

- Technically, note that the result is highly non-perturbative (it stems from non-trivial saddle point configurations).

- The result of these level correlations is typical for random matrix theory, i.e. for the level statistics of

$$H_{RMT=SYK_2} = \frac{1}{\sqrt{N}} c_a^\dagger t_{ab} c_b - \mu c_a^\dagger c_a, \tag{182}$$

  with random intersite hopping $\langle t_{ab} \rangle_{\text{dis.}} = 0; \langle t_{ab} t_{ab}^* \rangle_{\text{dis.}} = t^2 \sim 1/\Delta^2$

- Note that in this case, the SCBA saddle point equation (14a), is

$$G(i\epsilon) = [i\epsilon + \mu - \Sigma(i\epsilon)] \tag{183a}$$

$$\Sigma(\tau) = t^2 G(\tau), \tag{183b}$$

  We will find that the self energy dominates over $i\epsilon$, so that effectively

$$\Sigma = t^2(\mu - \Sigma)^{-1} \Rightarrow \Sigma = \frac{\mu - \sqrt{\mu^2 - 4t^2}}{2} \tag{184}$$

  which indeed is stronger than $i\epsilon$ at small ϵ and leads to a semicircular density of states

$$\nu(\epsilon) = \frac{\sqrt{1 - \epsilon^2/(2t)^2}}{\pi t} \tag{185}$$

- There are generalizations of Eq. (182) to the other Wigner-Dyson classes. These random matrices were introduced by Wigner to make some predictions about the level spectra of complex nuclei.

- However, there is a question of what the effect of interactions would be which ultimately are strong in nuclei. This was first addressed by French & Wong, and Bohigas & Flores in the early 70s - in modern notation the Hamiltonain is

$$H_{SYK_4} = \frac{1}{\sqrt{2N}^{3/2}} \sum_{a,b,c,d} U_{a,b,c,d} c_a^\dagger c_b^\dagger c_c c_d, \tag{186}$$

  with $\langle U_{a_1,a_2,a_3,a_4} \rangle_{\text{dis}} = 0, \langle U_{a_1,a_2,a_3,a_4} U^*_{a_1',a_2',a_3',a_4'} \rangle_{\text{dis}} = U^2 \prod_{i=1}^4 \delta_{a_i,a_i'}$. This model was revisted much later by Sachdev and Ye (early 90's), Georges, Parcollet and Sachdev (around 2000) and by Kitaev (mid 2010s) as a toy model of

  - a non-Fermi liquid
  - a tractable AdS/CFT duality
  - and of many-body quantum chaos

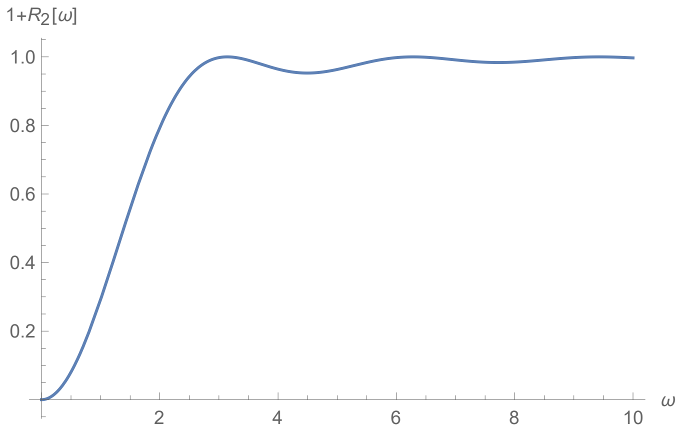


Figure 15: Normalized density of states correlator $\Delta^2 \langle \rho(\epsilon + \omega/2)\rho(\epsilon - \omega/2) \rangle = R_2(\omega) + 1$.

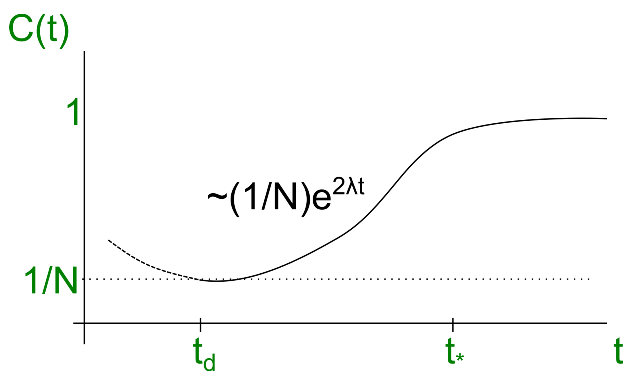


Figure 16: Schematic behavior of the funcion $C(t)$.

6 What is quantum chaos? Definitions

In classical physics:

- *Chaos:* Exponential sensitivity of phase-space trajectory to starting values. A typical definition for the Lyapunov exponent describing this sensitivity is
$$\max_{\mu,\nu=x_1,\ldots x_d} \left| \frac{\partial q_\mu(t)}{\partial q_\nu(0)} \right| \sim e^{\lambda t}. \tag{187}$$
Note that
$$\frac{\partial q_\mu(t)}{\partial q_\nu(0)} = \sum_{\rho=1}^{d} \frac{\partial q_\mu(t)}{\partial q_\rho(0)} \frac{\partial p_\nu(0)}{\partial p_\rho(0)} - \frac{\partial q_\mu(t)}{\partial p_\rho(0)} \frac{\partial p_\nu(0)}{\partial q_\rho(0)} = \{q_\mu(t), p_\nu(0)\}_{\text{P.B.}} \tag{188}$$

In quantum physics:

- There are several definitions for *Quantum Chaos*
  1. Bohigas-Giannoni-Schmidt conjecture : Any classically chaotic system displays random matrix statistics upon canonical quantization.

     ⇒: One definition: Random matrix statistics ≡ Quantum Chaos.

     This is related to the Eigenstate thermalization hypothesis (which is more a statement about ergodicity, but related to the first definition:)
     $$\langle E_i|V|E_j\rangle = \text{tr}\,[\rho V] + e^{-S(E)/2} f(E,\omega) R_{ij} \tag{189}$$
     for any sufficiently local operator V, eigenstates $|E_i\rangle$ at energy E and thermal density matrix defined by $E = \text{tr}\,[\rho H]$. $S = -\text{tr}\,[\rho \ln(\rho)]$ is the entropy, f a smooth function and R a random matrix.
  2. Quantum Lyapunov exponents and generalization of Eq. (187) : Operator spreading using canonical replacement of Poisson bracket by commutator
     $$\{q_\mu(t), p_\nu(0)\}_{\text{P.B.}} \to -i[\hat{q}_\mu(t), \hat{p}_\nu(0)], \tag{190}$$
     or more generally for arbitrary non-commuting observables
     $$\{A(t), B(0)\}_{\text{P.B.}} \to -i[\hat{A}(t), B(0)] \equiv \hat{O}(t). \tag{191}$$
     A natural generalization of the definition of the Lyapunov exponent would imply looking at$\langle O(t)\rangle = -i[\hat{q}_\mu(t), \hat{p}_\nu(0)]$, where $\langle \ldots \rangle$ could be the average w.r.t the ground state or w.r.t. a thermal ensemble with density matrix $\rho = e^{-\beta H}/\text{tr}\,[e^{-\beta H}]$. However, this not a useful definition, because

- Correlators will decay, not increase, as $t \to \infty$
- (this is related) the hermitian operator $O(t)$ is not positive definite and can thus have both positive and negatie contributions in the thermal average.

Instead, what is considered is

$$\tilde{C}(t) = -\langle [A(t), B(0)]^2 \rangle = -\mathrm{tr}\,[[A(t), B(0)]^2 \rho] \tag{192}$$

or its regularized version (thermofield double)

$$C(t) = -\mathrm{tr}\,[[A(t), B(0)]\rho^{-1/2}[A(t), B(0)]\rho^{-1/2}] \tag{193}$$

- Note that Eq. (193) contains terms $\langle A(t)B(0)A(t)B(0)\rangle$ which are out of time order correlators (OTOC).
- Typical behavior is presented in Fig. 16 for a system with N quantum degrees of freedom
  (a) At $t = 0$: $C(0)$ is a constant which can be ~ 1 (but it's not so important whether it's $\mathcal{O}(1)$ or small)
  (b) t_d, the dissipation time (or quasiparticle relaxation rate). At this time, two point correlators have died out, and $C(t_d) \sim 1/N$.
  (c) $t_d < t < t_*$: $C(t) \sim e^{2\lambda_{A,B}t}/N$ (This is the most interesting regime and defines the Lyapunov exponent.)
  (d) t_*, the scrambling time (or Ehrenfest time in the case of quantized single particle classical chaotic systems). $C(t)$ saturates beyond t_*
  The word scrambling is a quantum information theoretical word and refers to the following Gedanken-quench: Assume you have a quantum system with large number N of degrees of freedom. Even in a pure state, due to ETH, the entanglement entropy of a small subsystem can be assumed to be maximal, i.e. the information is smeared (scrambeled) over the entire system. Now perturb this small number of degrees of freedom and let the system evolve freely. The time after which this information from the local quench is scrambeled into the system is the scrambling time.
- A definition of quantum chaos is a regime of exponential growth for any pair A, B and the and the quantum Lyapunov exponent is thus $\lambda = \max_{V,W}[\lambda_{A,B}]$.

The interest in the quantum chaos and ergodicity is multifold

- One may ask the question, how quickly a system can thermalizes. This is closely related to the dissipation time t_d and the lifetime τ of local excitations (quasiparticles), which, amongst others, also appears in transport. It has been argued that there is a universal lower bound for this

$$\frac{1}{\tau} \lesssim \frac{1}{\tau_{\text{Planck}}} = \frac{T}{\hbar} \tag{194}$$

- One may further ask the question, about the rate of operator spreading λ, which is the question about the strength of quantum chaos. Again, there is a conjecture for an upper bound

$$\lambda \leq 2\pi T/\hbar \text{ (Maldacena, Shenker, Stanford bound on quantm chaos)} \tag{195}$$

This of course has an implication on the minimal scrambling time t_* (defined by $C(t_*) \sim 1$), and which thus is $t_* \sim \ln(N)/\lambda$ and therefore is bound from below (fast scrambling conjecture, saturated by some black holes.)

7 The SYK Model

We now return to Eq. (186) (and drop any quadratic terms of the kind of Eq. (182)). The model is named after Sachdev, Ye and Kitaev.

7.1 Self-consistent solution

The self-consistent equations in this case are given diagrammatically by Fig. 17, right.

$$G(i\epsilon) = [i\epsilon + \mu - \Sigma(i\epsilon)]^{-1} \tag{196a}$$

$$\Sigma(\tau) = -U^2 G(\tau)^2 G(-\tau), \tag{196b}$$

For simplicty, we set $\mu = 0$ (half-filling) and $T = 0$ to begin with. As before, we assume that $\Sigma(i\epsilon)$ dominates over $i\epsilon$ in the infrared, so that in a two-time index notation (of course all Green's function etc. only depend on the difference) and make the Ansatz

$$G(\tau) = -\#\text{sign}(\tau)|\tau|^{-2\Delta} \tag{197}$$

for long time scales $\tau U \gg 1$ which is equivalent to

$$G(i\epsilon) \sim |\epsilon|^{2\Delta - 1} \tag{198}$$

On the other hand

$$\Sigma(\tau) \sim U^2 \text{sign}(\tau)|\tau|^{-6\Delta} \Rightarrow \Sigma(i\epsilon) \sim -i\text{sign}(\epsilon)|\epsilon|^{6\Delta-1} \sim 1/G(i\epsilon). \tag{199}$$

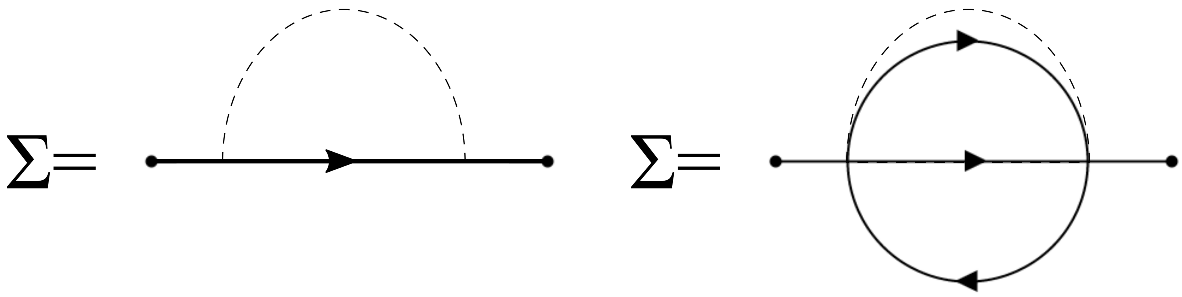


Figure 17: Left: Self energy for SYK-2 (i.e. random matrix theory, Eqs. (183) Right: Self energy equation for the SYK-4 model.

From comparing Eqs. (198) and (199) we find the condition

$$2\Delta - 1 = 1 - 6\Delta \Leftrightarrow \Delta = 1/4. \tag{200}$$

Comments:

- Indeed the self-energy $\Sigma \sim -i\text{sign}(\epsilon)|\epsilon|^{1/2}$ dominates over $i\epsilon$
- Contrary to the case of RMT, where $\Sigma = \text{const.}$, here the self energy and the associated decay rate are not only statistical but instead describe a proper non-Fermi liquid.
- At finite temperature, the propagator is (Parcollet and Georges)

  $$G(\tau) = -\text{sign}(\tau)\# \left(\frac{\pi T}{\sin(\pi T\tau)}\right)^{1/2}. \tag{201}$$

  Note the factor which smells like CFT.

7.2 Reparametrization symmetry

We now will uncover a low-energy reparametrization symmetry (i.e. a 1D conformal symmetry). In the IR, and written in time, the self-consistent equations are as follows

$$\int d\tau \Sigma(\tau_1, \tau) G(\tau, \tau_2) = -\delta(\tau_1 - \tau_2) \tag{202a}$$

$$\Sigma(\tau_1, \tau_2) = -U^2 G(\tau_1, \tau_2)^2 G(\tau_2, \tau_1). \tag{202b}$$

Let's use a reparametrization $\tau \mapsto f(\tau)$ for some $f(\tau)$ with $f'(\tau) > 0$ and assume the following transformation of Green's functions (i.e. Δ is the dimension of fermions)

$$G(\tau_1, \tau_2) \mapsto G(f(\tau_1), f(\tau_2)) f'(\tau_1)^{\Delta} f'(\tau_2)^{\Delta} \tag{203}$$

$$\Sigma(\tau_1, \tau_2) \mapsto \Sigma(f(\tau_1), f(\tau_2)) f'(\tau_1)^{3\Delta} f'(\tau_2)^{3\Delta} \tag{204}$$

Clearly the factor of 3 is to account for Eq. (202b). Eq. (202a) implies

$$\begin{aligned}\int df(\tau)\Sigma(f(\tau_1), f(\tau))G(f(\tau), f(\tau_2)) &= \int d\tau f'(\tau) \frac{\Sigma(\tau_1, \tau)}{f'(\tau_1)^{3\Delta} f'(\tau)^{3\Delta}} \frac{G(\tau, \tau_2)}{f'(\tau)^{\Delta} f'(\tau_2)^{\Delta}} \\ &\stackrel{\Delta=\frac{1}{4}}{=} -\frac{\delta(\tau_1 - \tau_2)}{f'(\tau_1)^{\Delta} f'(\tau_2)^{\Delta}} \\ &= -\delta(f(\tau_1) - f(\tau_2)).\end{aligned} \tag{205}$$

Comments

- Indeed, the finite temperature propagator can be obtained by a conformal map of the kind $\tau_{\text{line}} = \tan(\pi \tau_{\text{circle}} T)$, where $\tau_{\text{circle}} \in [-\beta/2, \beta/2)$.
- The emergent reparametrization symmetry, called *Diff* $(\mathbb{S}^1)$ on the circle, plays a role analogous to $U(2R)/U(R) \times U(R)$ for random matrix theory in class A, and will generate a set of (quasi-)softmodes.
- The reparametrization symmetry is explicitly broken at the UV (this is somewhat analogous to the ω term breaking the symmetry of soft-modes in the case of random matrix theory). The only actual UV transformations (on the line) are Möbius transformations, i.e. $SL(2, \mathbb{R})$

$$\tau \mapsto \frac{a + b\tau}{c + d\tau} \tag{206}$$

7.3 Effective action

We now look for an action which is somewhat analogous to Eq. (182) and accounts for fluctuations of the quasi-soft modes.

The fluctuations are best parametrized in terms of arbitrary $f(\tau)$ and the effective action should have the property $S_{\text{eff}}(\frac{a+b\tau}{c+d\tau}) = 0$.

The simplest action which suffices this condition is

$$S_{\text{eff}} = -\frac{N\gamma}{4\pi^2} \int_0^{\beta} d\tau \text{Sch}\left(\tan(\pi T f(\tau)), \tau\right), \tag{207}$$

where $\mathrm{Sch}\,(g,\tau) = g'''/g' - \frac{3}{2}(g''/g')^2$.

Comments:

- It is a nearly trivial Mathematica excercise to explicitly check $\mathrm{Sch}\left(\frac{a+b\tau}{c+d\tau},\tau\right) = 0$.
- This action can also be derived using a series of Hubbard-Stratonovich decouplings and expansion about the saddle point. The parameter γ is then obtained to be $\#/U$.
- The functional integral only runs over the massive modes (it excludes the Möbius transformations). The result leads to the entropy near zero temperature (Georges-Parcollet-Sachdev PRB 2001)

  $$S|_{T\to 0} = N\left(s_0 + \gamma T + \dots\right) \tag{208}$$

  Clearly γ is the specific heat. More counterintuitively, $s_0 = \frac{G}{\pi} + \frac{\ln(2)}{4} \approx 0.46$ is the zero temperature entropy (G is the Catalan constant - s_0 is strongly related to the Bekenstein-Hawking entropy of black holes). Note that s_0 does not stem from a degenerate ground state, but from a many-body level spacing $\Delta \sim e^{-N}$ (this is untypical, usual in many-body states, e.g. FL with quasiparticles, the level spacing above the ground state is $1/N$, where N in this case is system size).
- Actually, in the present case of the complex SYK, there are additional soft modes corresponding to the U(1) gauge symmetry.

7.4 OTOC, quantum chaos, scrambling.

In the regime $1 \ll \tau U < \beta U \ll N$ (i.e. requiring intermediate temperatures $J/N \ll T \ll J$) one may approximate the integration over the quasi-Goldstone modes by a Gaussian integral. We use this to calculate the operator spreading

$$C(t) = -\langle [c^\dagger(t), c(0)]^2 \rangle. \tag{209}$$

These operators are rewritten in terms of $f(\tau)$ near the saddlepoint and ultimately one finds after a rather tedious integral over the soft modes that

$$C(\tau) \sim \frac{\beta U}{N}\frac{\pi}{2}\sin\left(2\pi T\tau\right) \to e^{2\pi T t}/N. \tag{210}$$

This demonstrates that the SYK model saturates the conjectured bound on quantum chaos.

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